Math 241B: Functional Analysis
Winter, 2007

Homework Assignment 3

Due Wednesday, March 14

1. Let Ω be a nonempty, open, and bounded subset of $\mathbb{R}^n$. Let $u_k \in L^2(\Omega)$ $(k = 1, \dots)$ and $u \in L^2(\Omega)$. Assume that $\sup_{k\ge 1} \|u_k\|_{L^2(\Omega)} < \infty$. Prove that the following are equivalent:
   (1) $u_k \rightharpoonup u$ in $L^2(\Omega)$;
   (2) For any Lebesgue measurable subset $\omega \subseteq \Omega$, $\lim_{k\to\infty} \int_\omega u_k \, dx = \int_\omega u \, dx$;
   (3) For any n-dimensional cube $\omega \subseteq \Omega$, $\lim_{k\to\infty} \int_\omega u_k \, dx = \int_\omega u \, dx$.
2. Let E be a nonempty, bounded, Lebesgue measurable subset of $\mathbb{R}^n$. Prove that $u_k \to u$ in $L^1(E)$ if and only both $u_k \rightharpoonup u$ in $L^1(E)$ and $u_k \to u$ in measure.
3. Let Ω be a nonempty, open subset of $\mathbb{R}^n$. Let $\{K_i\}_{i=1}^\infty$ be a sequence of nonempty compact subsets of $\mathbb{R}^n$ such that $K_i \uparrow \Omega$. For each $i \ge 1$, let τ_i denote the locally convex, Hausdorff topology of $C^\infty_{K_i}(\Omega) := \{\phi \in C^\infty_c(\Omega) : \overline{\operatorname{supp} \phi} \subseteq K_i\}$, generated by the family of seminorms $\{p_{i,m}\}_{m=0}^\infty$, where $p_{i,m}(\phi) = \sup_{x\in K_i, |\alpha|\le m} |D^\alpha \phi(x)|$. Prove that $C^\infty_c(\Omega)$ and $\{(C^\infty_{K_i}(\Omega), \tau_i)\}_{i=1}^\infty$ form a strict inductive system.
4. An LCS is called a *Montel space* if it has the Heine-Borel property: every bounded subset is precompact. Prove the following:
   (1) A Montel space is complete;
   (2) A Montel space is reflexive;
   (3) If $(X, \tau) = \text{SI-}\lim_k (X_k, \tau_k)$ and (X_k, τ_k) $(k \ge 1)$ are all Montel spaces, then (X, τ) is also a Montel space.
5. Let Ω be a nonempty, open subset of $\mathbb{R}^n$. Prove the following:
   (1) $\phi_k \to \phi$ in $\mathcal{D}(\Omega)$ if and only if there exists a compact set $K \subset \Omega$ such that $\operatorname{supp}(\phi) \subseteq K$, $\operatorname{supp}(\phi_k) \subseteq K$ for all $k \ge 1$, and $D^\alpha \phi_k \to D^\alpha \phi$ uniformly on K for any multi-index α;
   (2) A subset $A \subseteq \mathcal{D}(\Omega)$ is bounded if and only if there exists a compact subset K of Ω such that
      (a) For any $\phi \in A$, $\operatorname{supp}(\phi) \subseteq K$; and
      (b) For any multi-index α, $\sup_{\phi\in A} \sup_{x\in K} |\partial^\alpha \phi(x)| < \infty$;
   (3) The space $\mathcal{D}(\Omega)$ is a Montel space. Hence, it is complete and reflexive.
6. Let Ω be a nonempty, open subset of $\mathbb{R}^n$.
   (1) For any integer $m \ge 0$, define $p_m : C^\infty_c(\Omega) \to \mathbb{R}$ by $p_m(\phi) = \sup_{x\in\Omega, |\alpha|\le m} |D^\alpha \phi(x)|$. Prove that the family of seminorms $\{p_m\}_{m=0}^\infty$ makes $C^\infty_c(\Omega)$ an LCS. Denote this topology by τ.
   (2) Let K be a nonempty compact subset of Ω. Prove that the restriction of the topology τ onto $C^\infty_K(\Omega)$ is exactly the topology of $C^\infty_K(\Omega)$ defined by the family of seminorms $\{p_{K,m}\}_{m=0}^\infty$ with $p_{K,m}(\phi) = \sup_{x\in K, |\alpha|\le m} |D^\alpha \phi(x)|$.
   (3) Is the topology τ the same as that of $\mathcal{D}(\Omega)$? Why?
7. Let Ω be a nonempty, open subset of $\mathbb{R}^n$ and $\{K_i\}_{i=1}^\infty$ a sequence of nonempty compact subsets of $\mathbb{R}^n$ such that $K_i \uparrow \Omega$. For each integer $i \ge 1$ and each integer $m \ge 0$,

define $p_{i,m} : C^\infty(\Omega) \to \mathbb{R}$ by $p_{i,m}(\phi) = \sup_{x\in K_i, |\alpha|\le m} |D^\alpha \phi(x)|$. With the family of seminorms $\{p_{i,m} : i \ge 1, m \ge 0\}$, the vector space $C^\infty(\Omega)$ is an LCS. This space is usually denoted by $\mathcal{E}(\Omega)$ and the topology is often called the *natural topology*. Prove the following:

(1) The space $\mathcal{E}(\Omega)$ is a Fréchet space;
(2) $\phi_j \to \phi$ in $\mathcal{E}(\Omega)$ if and only if $D^\alpha \phi_j \to D^\alpha \phi$ uniformly on any compact set $K \subset \Omega$ and for any multi-index α;
(3) The space $\mathcal{E}(\Omega)$ is a Montel space.

8. Let Ω be a nonempty, open subset of $\mathbb{R}^n$ and $T : \mathcal{D}(\Omega) \to \mathbb{F}$ be linear. Then $T \in \mathcal{D}'(\Omega)$ if and only if for any compact subset $K \subset \Omega$ there exist a constant $C > 0$ and an integer $m \ge 1$ such that

$$T(\phi) \le C \sup_{x\in K, |\alpha|\le m} |D^\alpha \phi(x)| \qquad \forall \phi \in C_c^\infty(\Omega) \text{ with } \mathrm{supp}\,(u) \subseteq K.$$

9. Let Ω be a nonempty open set of $\mathbb{R}^n$. Prove the following:
   (1) If $T_j \in \mathcal{D}'(\Omega)$ $(j \ge 1)$ and $T(\phi) = \lim_{j\to\infty} T_j(\phi)$ exists for any $\phi \in \mathcal{D}(\Omega)$, then $T \in \mathcal{D}'(\Omega)$;
   (2) If $T_j \in \mathcal{D}'(\Omega)$ $(j = 1, \dots)$ and $T \in \mathcal{D}'(\Omega)$ are such that $T_j(\phi) \to T(\phi)$ for any $\phi \in \mathcal{D}(\Omega)$, then $D^\alpha T_j(\phi) \to D^\alpha T(\phi)$ for any multi-index α and any $\phi \in \mathcal{D}(\Omega)$;
   (3) If $u_j \in L^1_{loc}(\Omega)$ $(j \ge 1)$ and $u_j|_K \to u|_K$ in $L^1(K)$ for any compact subset of Ω, then $D^\alpha T_{u_j} \to D^\alpha T_u$ in $\mathcal{D}'(\Omega)$ for any multi-index α.
10. Prove the following:
   (1) Let $-\infty < x_1 < \cdots < x_N < \infty$ and $f \in C^1(\mathbb{R}^1 \setminus \{x_1, \cdots, x_N\})$. Assume $f(x_j+0)$ and $f(x_j - 0)$ exist and are finite for each j $(1 \le j \le N)$. Then in $\mathcal{D}'(\mathbb{R}^1)$

$$\frac{d}{dx} T_f = T_{f'} + \sum_{j=1}^{N} [f(x_j + 0) - f(x_j - 0)]\, \delta_{x_j};$$

   (2) Let ω be a nonempty, bounded, open, and connected subset of $\mathbb{R}^n$ with a smooth boundary $\partial\omega$. Let $g : \mathbb{R}^n \to \mathbb{R}$ be such that $g|_{\overline{\omega}} \in C^1(\overline{\omega})$ and $g = 0$ outside $\overline{\omega}$. Prove that in $\mathcal{D}'(\mathbb{R}^n)$ and for each j $(1 \le j \le n)$

$$(\partial_j T_g)(\phi) = T_{\partial_j g}(\phi) + \int_{\partial\omega} \phi \nu_j \, dS \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^n),$$

   where ν_j is the jth component of the unit exterior normal ν at the boundary $\partial\omega$, and that

$$(\Delta T_g)(\phi) = T_{\Delta g}(\phi) + \int_{\partial\omega} \phi \nu_j \, dS \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^n).$$

   The derivative $\partial_j g$ and Δg are understood to be the locally integrable functions that are the same as the corresponding derivatives of g restricted onto ω but are zero outside $\overline{\omega}$.