3. Approximation and Numerical ODE

In this part, we assume that $a, b \in \mathbb{R}$ with $a < b$. We also denote by $\mathcal{P}_n$ the set of all polynomials of degree $\leq n$ for any integer $n \geq 0$.

Question 3.1.

(a) Prove for any $f \in C[a, b]$ that

$$\lim_{n\to\infty} \inf_{q_n \in \mathcal{P}_n} \max_{a \leq x \leq b} |f(x) - q_n(x)| = 0,$$

$$\lim_{n\to\infty} \inf_{q_n \in \mathcal{P}_n} \int_a^b [f(x) - q_n(x)]^2 dx = 0.$$

Proof. Denote $\|g\| = \max_{a \leq x \leq b} |g(x)|$ for any $g \in C[a, b]$. Since $\mathcal{P}_0 \subset \cdots \subset \mathcal{P}_n \subset \cdots$, we have $E_0(f) \geq \cdots \geq E_n(f) \geq \cdots$. Let $\varepsilon > 0$. By the first Weierstrass approximation theorem, there exists $p \in \mathcal{P}$ such that

$$\|f - p\| < \varepsilon.$$

Assume $p \in \mathcal{P}_N$ for some $N \geq 0$. Then for any $n \geq N$

$$0 \leq E_n(f) \leq E_N(f) \leq \|f - p\| < \varepsilon.$$

Therefore,

$$\lim_{n\to\infty} \inf_{q_n \in \mathcal{P}_n} \max_{a \leq x \leq b} |f(x) - q_n(x)| = 0.$$

Let $p_n \in \mathcal{P}_n$ be the best uniform approximation of f in $\mathcal{P}_n$. Then

$$\begin{aligned} 0 &\leq \inf_{q_n \in \mathcal{P}_n} \int_a^b [f(x) - q_n(x)]^2 dx \leq \int_a^b [f(x) - p_n(x)]^2 dx \\ &\leq \int_a^b \|f - p_n\|^2 dx = [E_n(f)]^2 (b - a) \to 0. \quad \text{as } n \to \infty. \qquad \textbf{Q.E.D.} \end{aligned}$$

(b) Let $p_2 \in \mathcal{P}_2$ be the best uniform approximation in $\mathcal{P}_2$ of the function $g(x) = x^3 - 2x^2 + 1$ with respect to the $C[-1, 1]$-norm. What is the value of $p_2(1)$? Justify your answer.

Solution. Let $\widetilde{T}_3(x) = 2^{-2} T_3(x) = (1/4) \cos(3 \arccos x)$. Then $\widetilde{T}_3(x) = x^3 - q_2(x)$ and $q_2 \in \mathcal{P}_2$ is the best uniform approximation of x^3 in $\mathcal{P}_2$ with respect to the $C[-1, 1]$-norm. Let $p_2(x) = q_2(x) - 2x^2 + 1 \in \mathcal{P}_2$. For any $r_2 \in \mathcal{P}_2$, we have $r_2(x) + 2x^2 - 1 \in \mathcal{P}_2$. Hence

$$\|g - p_2\| = \|x^3 - q_2(x)\| \leq \|x^3 - [r_2(x) + 2x^2 - 1]\| = \|g(x) - r_2(x)\|.$$

Thus $p_2(x) = q_2(x) - 2x^2 + 1$ is the best uniform approximation of $g(x) = x^3 - 2x^2 + 1$ in $\mathcal{P}_2$ with respect to the $C[-1, 1]$-norm. Now, since

$$q_2(x) = x^3 - \widetilde{T}_3(x) = x^3 - \frac{1}{4}\cos(3\arccos x),$$

we have

$$p_2(1) = q_2(1) - 1 = 1 - \frac{1}{4}\cos 0 - 1 = -\frac{1}{4}.$$

(c) Let $Q_0, \ldots, Q_n, \ldots$ be orthogonal polynomials in $L^2[a, b]$. Fix $n \geq 1$. Prove that Q_n has n simple roots in $[a, b]$.

Proof. Fix an integer $n \geq 1$. By the orthogonality, $\int_a^b Q_n(x)dx = 0$. Hence, Q_n changes its sign in (a, b) at least once. If $n = 1$, this implies that Q_1 has exactly one root. Consider $n \geq 2$. Suppose Q_n changes its sign in (a, b) only k times with $1 \leq k \leq n - 1$ at $x_1, \ldots, x_k$ with $a < x_1 < \cdots < x_k < b$. Define

$$p(x) = (x - x_1) \cdots (x - x_k).$$

Clearly, $p \in \mathcal{P}_k \subseteq \mathcal{P}_{n-1}$. Moreover, both Q_n and p change their signs only at $x_1, \ldots, x_k$. Thus,

$$\langle Q_n, p \rangle = \int_a^b Q_n(x)p(x)dx \neq 0.$$

This contradicts the fact that $\langle Q_n, q \rangle = 0$ for any $q \in \mathcal{P}_{n-1}$. Hence, $k \geq n$. But $Q_n \in \mathcal{P}_n$ can have at most n roots. Thus, $k = n$, and Q_n has exactly n simple roots in (a, b). **Q.E.D.**

Question 3.2.

(a) Find out the degree of precision of the numerical quadrature

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)[f(a)+f(b)] - \frac{1}{12}(b-a)^2[f'(b)-f'(a)] \qquad \forall f \in C^1[a, b].$$

Solution. The degree of precision is $m = 3$. To show this, we rewrite

$$f'(b) - f'(a) = \int_a^b f''(x)dx$$

which vanishes for any $f \in \mathcal{P}_1$. Notice that the first part $\frac{1}{2}(b - a)[f(a) + f(b)]$ is the value of the trapezoidal rule which is exact for any $p \in \mathcal{P}_1$. Thus, the given formula is exact for any $p \in \mathcal{P}_1$.

We now check with $\mathcal{P}_k$ for $k \geq 2$. For $k = 2$, we check with $f(x) = (x-a)^2$. We have

$$\int_a^b (x-a)^2 dx = \frac{1}{3}(b-a)^3$$

and

$$\frac{1}{2}(b-a)(b-a)^2 - \frac{1}{12}(b-a)^2 2(b-a) = \frac{1}{3}(b-a)^3.$$

Hence, it is exact.

For $k = 3$, we check with $f(x) = (x-c)^3$ with $c = (a+b)/2$. We have

$$\int_a^b (x-c)^3 dx = 0,$$
$$\frac{1}{2}(b-a)[(a-c)^3 + (b-c)^3] - \frac{1}{12}(b-a)^2 3[(b-c)^2 - (a-c)^2] = 0.$$

Hence, it is exact.

For $k = 4$, we try $f(x) = (x-a)^4$. We have

$$\int_a^b (x-a)^4 dx = \frac{1}{5}(b-a)^5$$
$$\frac{1}{2}(b-a)(b-a)^4 - \frac{1}{12}(b-a)^2 4(b-a)^3 = \frac{1}{6}(b-a)^5.$$

They are different. Therefore, the degree of precision is $m = 3$.

(b) Consider a sequence of interpolatory numerical integration formulas

$$\int_a^b f(x)\, dx \approx \sum_{k=1}^n A_k^{(n)} f\left(x_k^{(n)}\right), \qquad n = 1, \ldots.$$

Suppose all the coefficients $A_k^{(n)}$ $(k = 1, \cdots, n;\ n = 1, \cdots)$ are positive. Prove that

$$\lim_{n\to\infty} \sum_{k=1}^n A_k^{(n)} f\left(x_k^{(n)}\right) = \int_a^b f(x)\, dx \qquad \forall f \in C[a,b].$$

Proof. Let $L_{n-1} : C[a,b] \to \mathcal{P}_{n-1}$ be the Lagrange interpolation operator associated with $x_1^{(n)}, \ldots, x_n^{(n)}$. Fix $f \in C[a,b]$. For each $n \geq 1$, let $p_{n-1} \in \mathcal{P}_{n-1}$ be the best uniform approximation of f in $\mathcal{P}_{n-1}$. We have

$$E_{n-1}(f) := \|p_{n-1} - f\| = \min_{q \in \mathcal{P}_{n-1}} \|f - q\| \to 0 \qquad \text{as } n \to \infty,$$

where the norm is $C[a,b]$-norm. Since the quadrature is interpolatory with n points, it has degree of precision $\geq n-1$. Therefore,

$$
\begin{aligned}
&\left|\sum_{k=1}^{n} A_k^{(n)} f\left(x_k^{(n)}\right) - \int_a^b f(x)\,dx\right| \\
&\qquad \left|\sum_{k=1}^{n} A_k^{(n)} (L_{n-1}f)\left(x_k^{(n)}\right) - \int_a^b f(x)\,dx\right| \\
&\qquad = \left|\int_a^b (L_{n-1}f)(x)dx - \int_a^b f(x)\,dx\right| \\
&\qquad = \left|\int_a^b [(L_{n-1}f)(x) - p_{n-1}(x)]dx - \int_a^b [f(x) - p_{n-1}(x)]\,dx\right| \\
&\qquad \leq \left|\int_a^b [L_{n-1}(f - p_{n-1})](x)dx\right| + \int_a^b |f(x) - p_{n-1}(x)|dx \\
&\qquad \leq \left|\sum_{k=1}^{n} A_k^{(n)} (f - p_{n-1})\left(x_k^{(n)}\right)\right| + (b-a)E_{n-1}(f) \\
&\qquad \leq \sum_{k=1}^{n} \left|A_k^{(n)}\right| \|f - p_{n-1}\| + (b-a)E_{n-1}(f) \\
&\qquad = \left(\sum_{k=1}^{n} A_k^{(n)}\right) \|f - p_{n-1}\| + (b-a)E_{n-1}(f) \\
&\qquad = 2(b-a)E_{n-1}(f) \\
&\qquad \to 0 \qquad \text{as } n \to \infty,
\end{aligned}
$$

where we used the fact that all $A_k^{(n)} > 0$ and

$$\sum_{k=1}^{n} A_k^{(n)} = \int_a^b dx = b - a. \qquad \textbf{Q.E.D.}$$