Math 270C: Numerical Mathematics
Spring quarter, 2007
Homework Assignment 2
Due Tuesday, April 17

1. Prove the following properties of the Chebyshev Polynomials of first kind

$$T_n(x) = \cos(n \arccos x), \qquad n = 0, \cdots.$$

   (a) *Expansion.*

$$T_n(x) = \frac{1}{2}\left[\left(x + \sqrt{x^2-1}\right)^n + \left(x - \sqrt{x^2-1}\right)^n\right] \qquad n = 0, \cdots.$$

   (b) *Recursion formula.*

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad n = 1, \cdots.$$

   (c) *Orthogonality.*

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n = 0, \\ \pi/2 & \text{if } m = n > 0. \end{cases}$$

   (d) *Differential equations.*

$$\left(1-x^2\right) T_n''(x) - xT_n'(x) + n^2 T_n(x) = 0, \qquad n = 0, \cdots.$$

   (e) For each integer $n \geq 0$, $T_n \in \mathcal{P}_n$. Moreover, if n is even (odd), then T_n is an even (odd) polynomial.

2. (Chebyshev) Let $n \geq 0$ be an integer and T_n the nth Chebyshev polynomial of first kind. Let $P \in \mathcal{P}_n$ satisfy that $|P(x)| \leq 1$ for all $x \in [-1, 1]$. Show that

$$|P(y)| \leq |T_n(y)| \qquad \forall y \notin [-1, 1].$$

   (Hint: Otherwise, set $\lambda = P(y)/T_n(y)$ and consider $Q(x) = \lambda T_n(x) - P(x)$.)

3. Let $f \in C[a, b]$ and define

$$\mu_n(f) = \int_a^b x^n f(x)\, dx, \qquad n = 0, \cdots.$$

   Show that $f(x) = 0$ for all $x \in [a, b]$ if and only $\mu_n(f) = 0$ for all $n = 0, \cdots$.

4. Let $f(x) = x^4$. Find the least squares approximation of f in $\mathcal{P}_1$ over $[0, 1]$.

5. Let $\tilde{\mathcal{P}}_n$ denote the set of all polynomials of degree n with the leading coefficient 1. Let $\{Q_n\}_{n=0}^{\infty}$ with each $Q_n \in \tilde{\mathcal{P}}_n$ be the orthogonal polynomial system in $L^2[a, b]$. Prove that

$$\|Q_n\| = \min_{q \in \tilde{\mathcal{P}}_n} \|q\| \qquad \forall n \geq 0.$$