Math 270C: Numerical Mathematics
Spring quarter, 2007

Homework Assignment 3
Due Tuesday, May 1

1. Let
$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[(x^2 - 1)^n \right], \qquad n = 0, \cdots .$$
   (a) Show by the definition that $P_n \in \mathcal{P}_n$ for each n.
   (b) Show by Rolle's Theorem that, for $n \geq 1$, each P_n has n simple roots in $(-1, 1)$.
   (c) Directly verify for any $m \geq 0$ and $n \geq 0$ that
$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \begin{cases} 0 & \text{if } m \neq n, \\ 2/(2n+1) & \text{if } m = n. \end{cases}$$
2. Let $x_0 = 2$, $x_1 = 3$, $x_2 = 5$, $x_3 = 6$ and $y_0 = 5$, $y_1 = 2$, $y_2 = 3$, $y_3 = 4$. Let $p \in \mathcal{P}_3$ be the unique polynomial that interpolates y_j at x_j $(j = 0, 1, 2, 3)$.
   (a) Find the Lagrange form of p.
   (b) Find the Newton form of p.
3. Let $f(x) = x^4 - x^2 + 17x + 1$. Let $p \in \mathcal{P}_{20}$ interpolates f at $x_j = 2^j$ $(j = 0, \cdots, 20)$. Compute $p(0)$.
4. Let $x_0, \cdots, x_n$ be $n+1$ distinct real numbers. Let $l_1(x), \ldots, l_n$ be the associated basic Lagrange polynomials. Show that
$$\sum_{j=0}^{n} (x - x_j)^k\, l_j(x) = 0 \qquad \forall k = 1, \cdots, n.$$
5. Let $n \geq 1$ be an integer. Let $x_0, \cdots, x_n$ be $n+1$ distinct real numbers and $y_0, \cdots, y_n$ be $n+1$ real numbers. Suppose that $p_{n-1} \in \mathcal{P}_{n-1}$ interpolates y_j at x_j for $j = 0, \cdots, n-1$ and $q_{n-1} \in \mathcal{P}_{n-1}$ interpolates y_j at x_j for $j = 1, \cdots, n$. Define
$$p_n(x) = \frac{(x - x_0) q_{n-1}(x) - (x - x_n) p_{n-1}(x)}{x_n - x_0}.$$
   Show that $p_n \in \mathcal{P}_n$ and that p_n interpolates y_j at x_j for $j = 0, \cdots, n$.
6. Recall for $n \geq 1$ that the Chebyshev polynomial $T_n(x)$ has n distinct roots $x_j = \cos\theta_j$ with $\theta_j = (2j-1)\pi/2n$ $(j = 1, \cdots, n)$. Denote by $L_{n-1} : C[-1, 1] \to \mathcal{P}_{n-1}$ the Lagrange interpolation operator associated with $x_1, \ldots, x_n$. Show that
$$(L_{n-1} f)(x) = \frac{1}{n} \sum_{j=1}^{n} f(x_j) \frac{(-1)^{j-1} \sin\theta_j T_n(x)}{x - x_j} \qquad \forall f \in C[-1, 1].$$
7. Let $Q_n \in \mathcal{P}_n$ $(n = 0, 1, \cdots)$ be orthonormal polynomials in $L^2_\rho[a, b]$. Fix $n \geq 2$. Let $x_1, \cdots, x_n$ be the n distinct roots of $Q_n(x)$ in (a, b), and $l_1, \cdots, l_n$ be the associated basic Lagrange polynomials.
   (a) Prove that $l_1(x), \cdots, l_n(x)$ are orthogonal with respect to the inner product in $L^2_\rho[a, b]$.
   (b) Prove the identity
$$\sum_{j=1}^{n} \int_a^b \rho(x) \left[l_j(x) \right]^2 dx = \int_a^b \rho(x)\, dx.$$