Math 270C: Numerical Mathematics
Spring quarter, 2007
Homework Assignment 5
Due Thursday, May 17

1. Consider the trapezoidal formula
$$\int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)\,[f(a)+f(b)]\,.$$
   (a) Show that the degree of precision of the formula is $m = 1$.
   (b) Calculate the Peano kernel of the formula and show that it does not change sign in $[a, b]$.
   (c) Let $f \in C^2[a, b]$. Show that there exists $\xi \in (a, b)$ such that
$$\int_a^b f(x)\,dx - \frac{1}{2}(b-a)\,[f(a)+f(b)] = -\frac{1}{12}(b-a)^3 f''(\xi).$$
   (d) Let $N \geq 1$ be an integer, $h = (b-a)/N$, and $x_j = a + jh$, $j = 0, \cdots, N$. Prove
$$\int_a^b f(x)\,dx - \left\{ \frac{h}{2}\,[f(a)+f(b)] + h\sum_{j=1}^{N-1} f(x_j) \right\} = -\frac{(b-a)f''(\eta)}{12}h^2 \qquad \forall f \in C^2[a,b],$$
   where $\eta \in (a, b)$ depends on f.
   (e) Find an integer $N \geq 1$, as small as possible, so that $\left|\int_0^1 e^x dx - T_N\right| \leq 10^{-12}$, where T_N is the numerical integration value (without round-off error) of the function e^x over $[0, 1]$ using the composite trapezoidal rule with N subintervals of equal length.
2. Let $\{Q_n\}_{n=0}^{\infty}$ be a system of orthogonal polynomials in $L^2_\rho[a, b]$, where ρ is a weight function on $[a, b]$. Fix $n \geq 1$. Let $x_1, \cdots, x_n$ be the n distinct roots of Q_n in (a, b). Let
$$\int_a^b \rho(x) f(x)\,dx \approx \sum_{j=1}^{n} A_j f(x_j)$$
   be the corresponding weighted Gaussian quadrature, i.e., the constants A_j are given by $A_j = \int_a^b \rho(x) l_j(x)\,dx$ $(j = 1, \ldots, n)$, where $l_j \in \mathcal{P}_{n-1} (j = 1, \ldots, n)$ are the Lagrange basis polynomials. Prove the following:
   (a) For all $k = 1, \ldots, 2n-1$,
$$\sum_{j=1}^{n} A_j Q_k(x_j) = 0;$$
   (b) For all $j = 1, \ldots, n$,
$$A_j = \int_a^b \rho(x)[l_j(x)]^2 dx > 0;$$
   (c) Fny any $f \in C[a, b]$,
$$\left| \int_a^b \rho(x) f(x)\,dx - \sum_{j=1}^{n} A_j f(x_j) \right| \leq 2\left(\int_a^b \rho(x)\,dx\right) \min_{q \in \mathcal{P}_{2n-1}} \|f - q\|_{C[a,b]}.$$
3. Find the value of all x_1, x_2, x_3 and A_1, A_2, A_3 of the Gaussian quadrature
$$\int_{-1}^{1} f(x)\,dx \approx A_1 f(x_1) + A_2 f(x_2) + A_3 f(x_3).$$