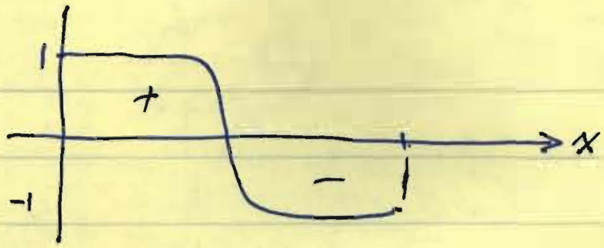
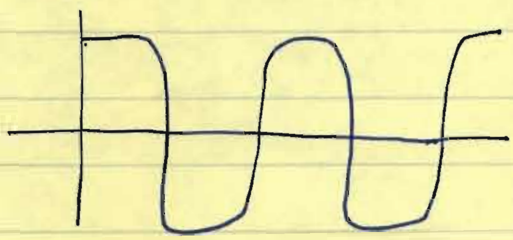




critical pts
but not
global minimizer.

3 Some remarks

- Boundary conditions
- Solution regularity
- non-uniqueness of solutions
- Newton iterations for nonlinear problems.
- Interface problems.

3.1 Boundary conditions ~~essential B.C.~~

(1) Dirichlet (or essential) B.C.

$u = u_0$ on $\partial\Omega$.

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

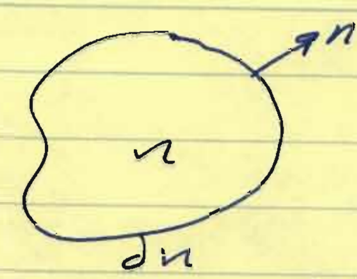
$$\min_{u = u_0 \text{ on } \partial\Omega} I[u]. \qquad I[u] = \int_\Omega \left[\tfrac{1}{2}|\nabla u|^2 - fu\right] dV$$

↑ enforced.

$$\left.\frac{d}{dt}\right|_{t=0} I[u+tv] = 0 \qquad \forall v: \ v = 0 \text{ on } \partial\Omega.$$

② Neumann (or natural) B.C.

$$\begin{cases} \nabla\cdot\varepsilon(x)\nabla u = -f & \Omega \\ \varepsilon(x)\dfrac{\partial u}{\partial n} = \sigma & \partial\Omega \end{cases}$$



Notes

(a) Solutions not unique.

u is a solution $\Longrightarrow$ $u+1$ is also a solution.

But, unique up to an additive constant.

(b) A compatibility or Solvability condition

$$\int_\Omega f\,dV + \int_{\partial\Omega} \sigma\,dS = 0$$

— charge neutrality!

$$I[u] = \int_\Omega \left[\frac{\varepsilon(x)}{2}|\nabla u|^2 - fu\right]dV - \int_{\partial\Omega}\sigma u\,dS$$

Euler-Lagrange equation. v: ~~arbitrary~~ arbitrary

$$\delta I[u][v] = \frac{d}{dt}\Big|_{t=0} I[u+tv]$$

$$= \frac{d}{dt}\Big|_{t=0}\int_\Omega\left[\frac{\varepsilon(x)}{2}|\nabla u + t\nabla v|^2 - f(u+tv)\right]dV - \frac{d}{dt}\Big|_{t=0}\int_{\partial\Omega}\sigma(u+tv)$$

$$= \int_\Omega\left[\varepsilon(x)\nabla u\cdot\nabla v - fv\right]dV - \int_{\partial\Omega}\sigma v\,dS$$

assume u is smooth

$$= \int_\Omega\left(-\nabla\cdot\varepsilon(x)\nabla u - f\right)v\,dV + \int_{\partial\Omega}\left[\varepsilon\frac{\partial u}{\partial n} - \sigma\right]v\,dS.$$

Choose first v: $v=0$ on $\partial\Omega$.

$$\delta I[u][v]=0 \Longrightarrow \int_\Omega\left(-\nabla\cdot\varepsilon\nabla u - f\right)v\,dV = 0 \quad \forall v,\ v=0\ \partial\Omega$$

$$\Longrightarrow -\nabla\cdot\varepsilon\nabla u = f \quad \text{in } \Omega$$

Now, $\forall v$. $$\delta I[u][v] = \int_{\partial\Omega}\left[\varepsilon\frac{\partial u}{\partial n} - \sigma\right]v\,dS = 0 \quad \forall v.$$

$$\Longrightarrow \varepsilon\frac{\partial u}{\partial n} = \sigma \quad \text{on } \partial\Omega.$$

min. $I[u]$ $\qquad I[u] = \int_\Omega \left(\frac{\varepsilon}{2}|\nabla u|^2 - fu\right) dV - \int_{\partial\Omega} \sigma u \, dS$

Key: $I[u] \geq c_1 \|u\|_{H^1}^2 - c_2$.

$I[u] = I[u - \fint_\Omega u]$ $\qquad \fint_\Omega u = \frac{1}{|\Omega|}\int_\Omega u \, dV$

or $I[u] = I[u+c]$ $\quad \forall c = \text{const}$.

Check: $I[u+c] = \int_\Omega \left(\frac{\varepsilon}{2}|\nabla u|^2 - fu\right) dV - \int_{\partial\Omega} \sigma u \, dS$

$$\underbrace{- \int_\Omega fc \, dV - \int_{\partial\Omega} \sigma c \, dS}_{= c\left[\int_\Omega f + \int_{\partial\Omega} \sigma\right] = 0.}$$ compatibility

$\Rightarrow$ min $I[u]$

$u: \int_\Omega u \, dV = 0$.

Poincaré ineq $\Rightarrow$ bound!

③ Mixed (or Robin) B.C

$$\begin{cases} -\nabla \cdot \varepsilon \nabla u = f & \text{in } \Omega \\ \varepsilon \frac{\partial u}{\partial n} + bu = 0 & \text{on } \partial\Omega \end{cases}$$

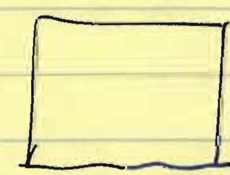
$\varepsilon \geq \varepsilon_0$. $\quad b \geq 0$.

$I[u] = \int_\Omega \left(\frac{\varepsilon}{2}|\nabla u|^2 - fu\right) dV - \int_{\partial\Omega} \frac{b}{2} u^2 \, dS$

④ Periodic B.C

$u(x + l_i e_i) = u(x)$.

$(l_1, \dots, l_3)$ - period.



$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

3.2 Solution regularity

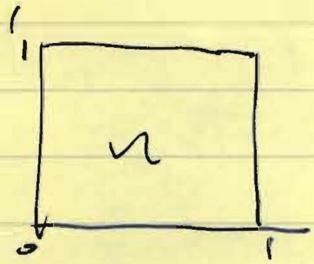
ODE: $\quad -u'' = f \qquad u' = -\int^x f(s)\,ds + c_1$

$\qquad\qquad\qquad\qquad u = \int \cdots$

u — smooth if f is.

PDE: regularity dep. on $\underbrace{\text{coeff.} + \text{right hand}}$ $+\ \Omega$:

Example



$\Delta u = 1 \quad \Omega$

$u = 0 \quad \partial\Omega$

claim $\quad u \notin C^2(\bar\Omega)$.

Proof $\quad$ HW ("To-do it at your wish.)

$(0,0) \qquad \frac{\partial u}{\partial x_1}(x_1, 0) = 0. \quad \frac{\partial^2 u}{\partial x_1^2}(x_1, 0) = 0. \quad \frac{\partial^2 u}{\partial x_1^2}(0,0) = 0$

similarly $\frac{\partial^2 u}{\partial x_2^2}(0,0) = 0. \implies \Delta u(0,0) = 0.$

$\Delta u = 1$ in Ω (not $\bar\Omega$). But, take limit!

Laverentive phenomenon

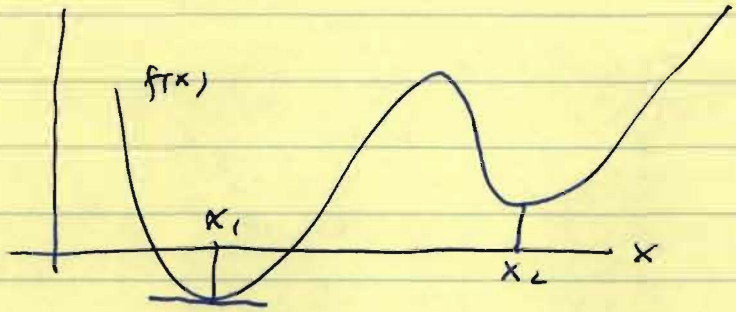
spaces (or sets) $\quad X \supseteq Y$

$I: X \to R. \qquad$ e.g. $\quad X = L^2(\Omega)$, $\quad Y = H^1(\Omega)$

$\inf_{u\in X} I[u], \quad \inf_{u \in Y} I[u]$

exist? $\quad$ same?

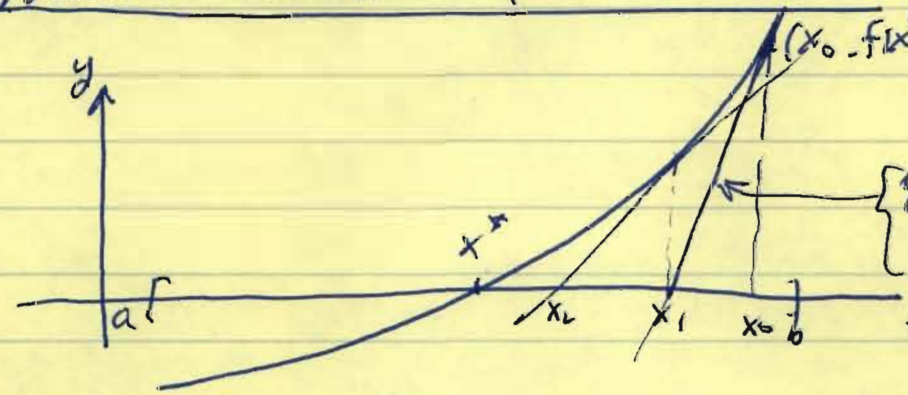
3.3 Non-uniqueness of solutions



$f'(x_1)=0$, $f'(x_2)=0$.

But: dynamics, time-dep. process $\overset{\text{often}}{\Longrightarrow}$ uniqueness.

3.4. Newton's iteration for nonlinear problems



$$\begin{cases} y = f'(x_0)(x-x_0)+f(x_0) \\ y=0 \end{cases} \Longrightarrow \boxed{x = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

Assume $f(x)=0$ has a sol'n $x^* \in [a,b]$.
f is smooth.
How to find x^* numerically?

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \qquad k=1,2,\dots$$

Instead of solving $\quad y = f(x)$
we solve $\quad y = f'(x_k)(x-x_k)+f(x_k)$

Monday. 4/4/2011.

Summary so far: Nonlinear Diffusion Equations.
- steady-state, time-dependent, variational principles
- interface problems, numerics

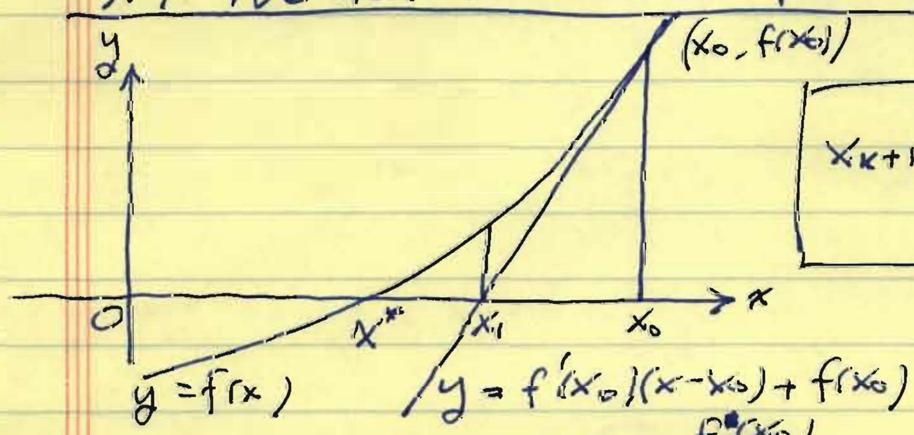
Finished:
- Examples
- Linear problems
- Nonlinear problems: variational principles.

In the middle of remarks

Then: The Poisson-Boltzmann Theory of continuum electrostatics

3. Remarks
3.1. Boundary conditions. (Done)
3.2 Solution regularity (Done)
3.3 Non-uniqueness of solutions (Done)

3.4 Newton iterations for nonlinear problems



$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad (k = 0, 1, \cdots)}$$

quadratic convergence.

Fixed-point iteration: $x_* = g(x_*)$ (assumption)

$$x_{k+1} = g(x_k) \quad k = 0, 1, \cdots$$

$$x_{k+1} - x_* = g(x_k) - x_* = g(x_k) - g(x_*)$$

$$= g'(x_*)(x_k - x_*) + \frac{1}{2} g''(x_*)(x_k - x_*)^2 + \cdots$$

$$x_{k+1} - x_* \approx g'(x_*)(x_k - x_*)$$

If $|g'(x_*)| < 1$ then convergence. linear convergence.

Newton's iteration: $g(x) = x - \frac{f(x)}{f'(x)}$, $f(x_*) = 0$

$g'(x_*) = 0$. quadratic convergence!

Apply to solving nonlinear PDEs.

$$\begin{cases} \Delta u + F(u) = f & \text{in } \Omega \\ + \text{B.C.} \end{cases}$$

Let $N[u] = \Delta u + F(u) - f$

Given u_0. How to find u_1?

Formally. $\underbrace{N'(u_0)(u_1 - u_0)}_{\delta N(u_0)(u_1 - u_0)} + N(u_0) = 0$

<u>Definition</u>

$$\delta N(u)(v) = \frac{d}{dt}\Big|_{t=0} N(u + tv)$$

$$= \Delta v + \cancel{F'(u)v}\ \delta F(u)(v) \qquad = F'(u)(v)$$

$$\delta N(u_0)(u_1 - u_0) = \Delta u_1 - \Delta u_0 + F'(u_0)(u_1 - u_0)$$

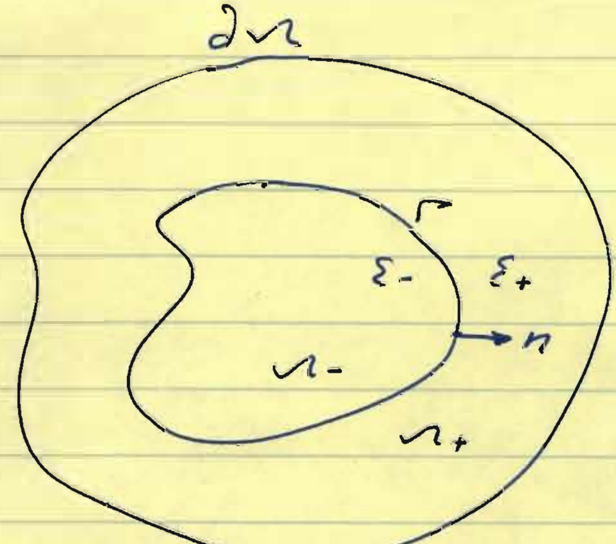
$$N(u_0) = \Delta u_0 + F(u_0) - f$$

$$\Delta u_1 - \Delta u_0 + F'(u_0)(u_1 - u_0) + \Delta u_0 + F(u_0) - f = 0$$

$$\begin{cases} \Delta u_1 + F'(u_0)(u_1 - u_0) = f - F(u_0) & \Omega \\ \text{B.C.} & \partial\Omega \end{cases}$$

$$\boxed{\text{General: } \Delta u_{k+1} + F'(u_k) u_{k+1} = f + F'(u_k) u_k - F(u_k)}$$

3.5. An equivalent formulation of elliptic interface problems



$$\begin{cases} -\nabla\cdot\varepsilon(x)\nabla u = f & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

An equivalent formulation

$$\begin{cases} -\varepsilon_-\Delta u = f & \text{in } \Omega_- \\ -\varepsilon_+\Delta u = f & \text{in } \Omega_+ \\ [\![u]\!]_\Gamma = 0 \\ [\![\varepsilon\frac{\partial u}{\partial n}]\!] = 0 \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

$$\varepsilon(x) = \begin{cases} \varepsilon_- & \text{if } x\in\Omega_- \\ \varepsilon_+ & \text{if } x\in\Omega_+ \end{cases}$$

Notation $[\![a]\!] = a|_{\Omega_+} - a|_{\Omega_-}$.

Why equivalent?

(10) $\xrightarrow{u=u_0 \text{ on } \partial\Omega:}$ $\displaystyle\int_\Omega \varepsilon(x)\nabla u\cdot\nabla\varphi\, dV = \int_\Omega f\varphi\, dV \quad \forall\varphi\in H_0^1(\Omega)$

Choose $\varphi\in H_0^1(\Omega)$ s.t. $\operatorname{supp}\phi\subset\Omega_-$.

$$\int_{\Omega_-}\varepsilon_-\nabla u\cdot\nabla\varphi\, dV = \int_{\Omega_-} f\varphi\, dV \quad \forall\varphi\in C_0^1(\Omega_-).$$

This means that

$$-\varepsilon_-\Delta u = f \quad \text{in } \Omega_-.$$

Similarly $-\varepsilon_+\Delta u = f$ in Ω_+.

By (10), $\displaystyle\int_\Omega f\varphi\, dV = \int_{\Omega_-} f\varphi\, dV + \int_{\Omega_+} f\varphi\, dV$

$$\int_\Omega \varepsilon(x)\nabla u\cdot\nabla\varphi\, dV = \int_{\Omega_-}\varepsilon_-\nabla u\cdot\nabla\varphi\, dV + \int_{\Omega_+}\varepsilon_+\nabla u\cdot\nabla\varphi\, dV$$

$$= -\int_{\Omega_-}\varepsilon_-\Delta u\,\varphi\, dV + \int_\Gamma \varepsilon_-\frac{\partial u}{\partial n}\varphi\, dS$$
$$-\int_{\Omega_+}\varepsilon_+\Delta u\,\varphi\, dV - \int_\Gamma \varepsilon_+\frac{\partial u}{\partial n}\varphi\, dS$$

Hence
$$\int_\Gamma \left[\varepsilon_+ \frac{\partial u^+}{\partial n} - \varepsilon_- \frac{\partial u^-}{\partial n}\right]\varphi \, dS = 0. \quad \forall \varphi$$
$$\left[\!\left[\varepsilon \frac{\partial u}{\partial n} \right]\!\right]_\Gamma = 0$$

What about $[\![u]\!]_\Gamma = 0$? This is the assumption for a solution:

$$u \in H^1(\Omega), \quad u = u_0 \text{ on } \partial\Omega$$
$$\int_\Omega \varepsilon \nabla u \cdot \nabla \varphi \, dV = \int_\Omega f \varphi \, dV \quad \forall \varphi \in H_0^1(\Omega)$$

The other direction of proof.

Use. Theorem 1.2 in R. Teman: Navier-Stokes Equations 1984.

$$\left.\begin{array}{l} a \in L^2(\Omega) \\ \nabla \cdot a \in L^2(\Omega) \end{array}\right\} \Longrightarrow a \cdot n \in H^{-\frac{1}{2}}(\Gamma)$$

also, The divergence theorem holds true.