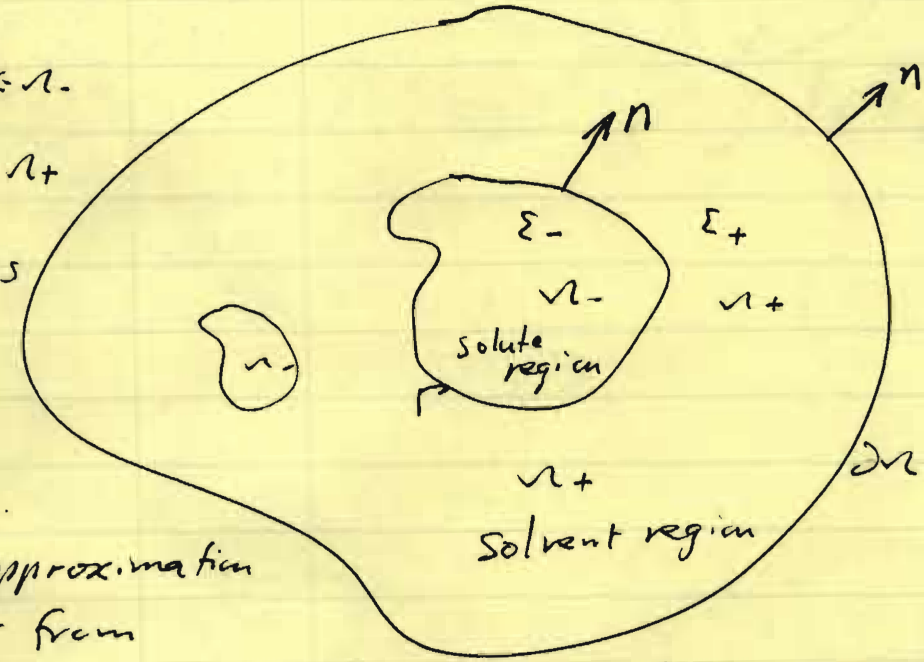
Dielectric Boundary Forces (DBF) in molecular solvation.



Notation

⊙ $\varepsilon(x) = \varepsilon_\Gamma(x) = \begin{cases} \varepsilon_- & \text{if } x \in \Omega_- \\ \varepsilon_+ & \text{if } x \in \Omega_+ \end{cases}$

often writ it as

$\varepsilon_\Gamma = \varepsilon_\Gamma(x)$.

⊙ $f : \Omega \to \mathbb{R}$: fixed charge.

e.g. f is an approximation of point charges from solute particles inside Ω_-.

Ω: system region

⊙ $\chi_+ = \chi_{\Omega_+}$: the characteristic function of Ω_+.

Recall that the minimum electrostatic free energy is given by

$$G[\Gamma] = \int_\Omega \left[-\frac{\varepsilon_\Gamma}{2} |\nabla \psi|^2 + f\psi - \chi_+ B(\psi) \right] dx$$

Here

$\psi = \psi(x)$ is the electrostatic potential, the unique solution of the boundary-value problem of a PB-like equation

$$\begin{cases} \nabla \cdot \varepsilon_\Gamma \nabla \psi - \chi_+ B'(\psi) = -f & \Omega \\ \psi = g & \partial\Omega \end{cases}$$

↑ a given function (B.C. datum)

$-B'(\psi) = \sum_{j=1}^{M} q_j c_j(x)$: charge density from mobile ions in the solvent

Examples of $B(\psi)$.

In general: $B(\psi) > B(0) = 0$ if $\psi \neq 0$.
$B(\psi)$ convex. $B'(\psi) \begin{cases} > 0 & \text{if } \psi > 0 \\ < 0 & \text{if } \psi < 0 \\ = 0 & \text{if } \psi = 0 \end{cases}$

- $$B(\psi) = \beta^{-1} \sum_{j=1}^{M} c_j^{\infty} \left(e^{-\beta \varepsilon_j \psi} - 1 \right)$$

  — the classical PB

- $$B(\psi) = \beta^{-1} a^3 \left[1 + \log\left(1 + \sum_{j=1}^{M} a^3 c_j^{\infty} e^{-\beta \varepsilon_j \psi} \right) \right]$$

  — the generalized PB with a uniform ionic size $a_0 = a_1 = \cdots = a_M$.

- $$B(\psi) = -\int_0^{\psi} \sum_{j=1}^{M} q_j c_j(\phi)\, d\phi .$$

  $c_j(\phi)$: $\left(\frac{a_j}{a_0}\right)^3 \log(a_0^3 c_0) - \log(a_j^3 c_j) = \beta(\varepsilon_j \psi - \mu_j)$, $j = 1, \ldots, M$

  $$a_0^3 c_0 = 1 - \sum_{j=1}^{M} a_j^3 c_j .$$

  — non uniform ion sizes.

<u>Goal</u>: Define and calculate the dielectric boundary force: — variations of $G[\Gamma]$ w.r.t. the location change of Γ.

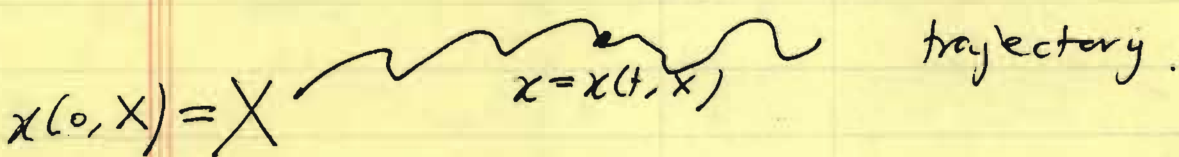
Given Γ: $\longrightarrow$ determine $\varepsilon_\Gamma \longrightarrow$ PBE: $\psi = \psi_\Gamma$
$\longrightarrow$ free energy $G[\Gamma]$.

Shape derivatives

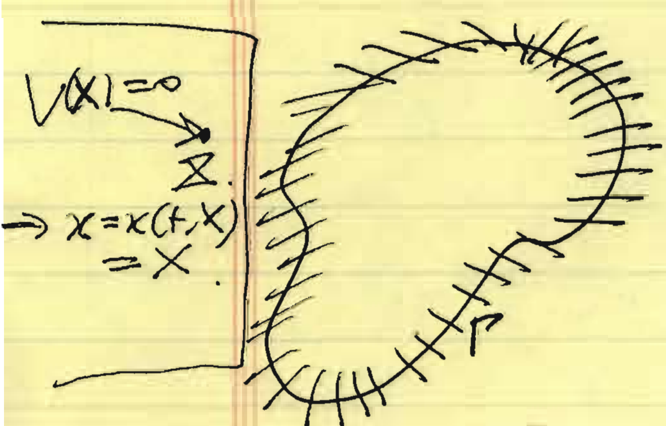
Perturbation of Γ locally. Let $V \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$

$$\begin{cases} \dfrac{dx(t,X)}{dt} = V(x(t,X)) & \forall t > 0 \text{ small} \\ x(0,X) = X & \forall X \in \mathbb{R}^3 \end{cases}$$

$x = x(t,X)$ is the flow determined by V.



For each $t \geq 0$. $x = x(t,X)$ ~~det~~ is a diffeomorphism $\mathbb{R}^3 \to \mathbb{R}^3$ 1-1, onto, differentiable.



$V = 0$ outside shaded region

or fix $z \in \Gamma$.
$V = 0$ outside a ball centered at z.

Local perturbations!

$\Gamma_t = \Gamma_t(V) = \{ x(t,X) : X \in \Gamma \}$.

Write: $T_t(X) = x(t,X)$.

<u>Definition</u> The shape derivative of the electrostatic free energy $G[\Gamma]$ in the direction of $V \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$ is

$$\delta_{\Gamma,V} G[\Gamma] = \frac{d}{dt}\Big|_{t=0} G[\Gamma_t(V)].$$

We will see that

unit normal along Γ

$$\delta_{\Gamma,V} G[\Gamma] = \int_\Gamma \boxed{\quad\quad} (V\cdot n)\, dS$$

We define the shape derivative in the direction n,

$$\delta_\Gamma G[\Gamma] : \Gamma \to \mathbb{R},$$

to be $\delta_\Gamma G[\Gamma] = \boxed{\quad\quad}$

DBF: $\quad F_n = -\delta_\Gamma G[\Gamma]$

~~Thm~~

<u>Thm.</u> $$\delta_\Gamma G[\Gamma] = \frac{\varepsilon_+}{2}|\nabla\psi^+|^2 - \frac{\varepsilon_-}{2}|\nabla\psi^-|^2 - \varepsilon_+|\nabla\psi^+\cdot n|^2 + \varepsilon_-|\nabla\psi^-\cdot n|^2 + B(\psi)$$

$$= \frac{1}{2}\left(\frac{1}{\varepsilon_-} - \frac{1}{\varepsilon_+}\right)|\varepsilon_\Gamma \nabla\psi\cdot n|^2 + \frac{1}{2}(\varepsilon_+ - \varepsilon_-)|(I - n\otimes n)\nabla\psi|^2 + B(\psi).$$

If $0 < \varepsilon_- < \varepsilon_+$ then $\delta_\Gamma G[\Gamma] > 0$ on Γ. Hence

$$F_n = -\delta_\Gamma G[\Gamma] < 0$$

Forces toward soluteS!

The Maxwell Stress Tensor

$$T = \varepsilon_p E \otimes E - \frac{\varepsilon_r}{2} |E|^2 I - \chi_+ B(\psi) I.$$

$E = -\nabla \psi$ ... the electric field

Easy to verify:

$$F_n = [\![n \cdot T n]\!] = n \cdot T^+ n - n \cdot T^- n$$

= jump of surface forces.

Recall $\begin{cases} \nabla \cdot \varepsilon_p \nabla \psi - \chi_+ B'(\psi) = -f & \text{in } \Omega \\ \psi = g & \text{on } \partial\Omega \end{cases}$ $(*)$

Define $G[\Gamma, \phi] = \int_\Omega \left[-\frac{\varepsilon_r}{2} |\nabla \phi|^2 + f\phi - \chi_+ B(\phi) \right] dV.$

Set $H^1_g(\Omega) = \{ u \in H^1(\Omega) : u = \psi_0 \text{ on } \partial\Omega \}.$

Thm. (1) $G[\Gamma, \cdot] : H^1_g(\Omega) \to \mathbb{R} \cup \{\pm\infty\}$ has a unique maximizer $\psi_0 \in H^1_g(\Omega)$.
Moreover, $\exists\, C_1 = C_1(\varepsilon_-, \varepsilon_+, f, \psi_0, B, \Omega) > 0$, not depending on Γ, such that

$$\|\psi_0\|_{H^1(\Omega)} + \|\psi_0\|_{L^\infty(\Omega)} \le C_1$$

(2) The maximizer ψ_0 is the unique solution to the boundary-value problem of the PB equation $(*)$. $\square$

Derivation of F_n:

See Bo Li, Xiaoliang Cheng, and Zhengfang Zhang, Dielectric Boundary Force in Molecular Solvation with the Poisson-Boltzmann Free Eenergy: A shape Derivative Approach, 2011 (submitted).

Outline of the derivation of the force F_n.

<u>Step 1</u> We write

$$G[\Gamma_t] = \max_{\phi \in H_g^1(\Omega)} G[\Gamma_t, \phi \circ T_t^{-1}].$$

$$\left(\text{same as } G[\Gamma_t, \phi], \text{ since } \phi \text{ ranges over } H_g^1(\Omega) \Longleftrightarrow \phi \circ T_t^{-1} \text{ ranges over } H_g^1(\Omega) \right)$$

Let $\phi \in H^1(\Omega) \cap L^\infty(\Omega)$, $t \geq 0$. Let

$$z(t, \phi) = G[\Gamma_t, \phi \circ T_t^{-1}].$$

By properties of $T_t(X)$ we have

$$z(t, \phi) = \int_\Omega \left[-\frac{\varepsilon_{\Gamma_t}}{2} |\nabla(\phi \circ T_t^{-1})|^2 + f\,(\phi \circ T_t^{-1}) - \chi_{t+} (B(\phi) \circ T_t^{-1}) \right] dx$$

$$\overset{x = T_t(X)}{=\!=} \int_\Omega \left[-\frac{\varepsilon_p}{2} A(t) \nabla\phi \cdot \nabla\phi + (f \circ T_t)\phi J_t - \chi_t B(\phi) J_t \right] dX$$

$$\left[\text{where } J_t(X) = \det \nabla T_t(X) \text{ and } A(t) = J_t (\nabla T_t)^{-1} (\nabla T_t)^{-T}. \right]$$

$$= \int_\Omega \left[-\frac{\varepsilon_p}{2} A'(t) \nabla\phi \cdot \nabla\phi + ((\nabla \cdot (fV)) \circ T_t)\phi J_t - \chi_t B(\phi) ((\nabla \cdot V) \circ T_t) J_t \right] dX.$$

Step 2. Let $\psi_t \in H_g^1(\Omega) \cap L^\infty(\Omega)$ maximize $G[\rho_t, \cdot]$.

We have

$$G[\rho, \psi_t] \le G[\rho, \psi_0] = G[\rho]$$

$$G[\rho_t, \psi_0 \circ T_t^{-1}] \le G[\rho_t, \psi_t] = G[\rho_t]$$

Hence

$$\frac{G[\rho_t, \psi_0 \circ T_t^{-1}] - G[\rho, \psi_0]}{t} \le \frac{G[\rho_t] - G[\rho]}{t} \le \frac{G[\rho_t, \psi_t] - G[\rho, \psi_t]}{t}.$$

Hence

$$\frac{z(t, \psi_0) - z(0, \psi_0)}{t} \le \frac{G[\rho_t] - G[\rho]}{t} \le \frac{z(t, \psi_t \circ T_t) - z(0, \psi_t \circ T_t)}{t}$$

$\Rightarrow \exists\, \xi(t), \eta(t) \in [0, t]$ (for each $t \in [0, \tau]$, $\tau > 0$: small)

such that

$$\partial_t z(\xi(t), \psi_0) \le \frac{G[\rho_t] - G[\rho]}{t} \le \partial_t z(\eta(t), \psi_t \circ T_t)$$

$$\forall t \in (0, \tau].$$

Step 3 We prove

$$\lim_{t \to 0} \partial_t z(\xi(t), \psi_0) = \partial_t z(0, \psi_0),$$

$$\lim_{t \to 0} \partial_t z(\eta(t), \psi_t \circ T_t) = \partial_t z(0, \psi_0).$$

Only prove the 2nd one. (The 1st one can be proved similarly.)

We have $A'(\eta(t)) \rightrightarrows A'(0)$ as $t \to 0$.

$J_{\eta(t)} \rightrightarrows J_0 = 1$

["$\rightrightarrows$" old Russian notation: uniform convergence!]

Consequently,

$$(\nabla\cdot(fV))\circ T_{\eta(t)} \longrightarrow \nabla\cdot(fV) \quad \text{in } L^2(\Omega)$$
$$(\nabla\cdot V)\circ T_{\eta(t)} \longrightarrow \nabla\cdot V \quad \text{in } L^2(\Omega).$$

We need: $\lim_{t\to 0}\|\psi_t\circ T_t - \psi_0\|_{H^1(\Omega)} = 0$ (*)

If (*) is true, then: [Notice that ψ_t is uniformly bdd in $L^\infty(\Omega)$ with resp. to $t\in[0,T]$.]

· $B(\psi_t\circ T_t) - B(\psi_0) = B'(\lambda(t))(\psi_t\circ T_t - \psi_0) \longrightarrow 0$ in $H^1(\Omega)$.

Hence

$$\begin{aligned}
&\partial_t z(\eta(t), \psi_t\circ T_t) \\
&= \int_\Omega \Big[-\frac{\varepsilon_p}{2} A'(\eta(t))\nabla(\psi_t\circ T_t)\cdot\nabla(\psi_t\circ T_t) \\
&\qquad + \big((\nabla(fV))\circ T_{\eta(t)}\big)(\psi_t\circ T_t)\, J_{\eta(t)} \\
&\qquad - \chi_+ B((\psi_t\circ T_t))\big((\nabla\cdot V)\circ T_{\eta(t)}\big) J_{\eta(t)} \Big] dX \\
&\xrightarrow{t\to 0} -\int_\Omega \Big[-\frac{\varepsilon_p}{2} A'(0)\nabla\psi_0\cdot\nabla\psi_0 + (\nabla\cdot(fV))\psi_0 \\
&\qquad - \chi_+ B(\psi_0)(\nabla\cdot V)\, dX \\
&= \partial_t z(0, \psi_0).
\end{aligned}$$

The (*) is proved mainly by ~~th~~ using the equation for ψ_0, and ψ_t, and the convexity of B.

Step 4. We now have

$$\delta_{\Gamma,V} G[\Gamma] = \frac{d}{dt}\Big|_{t=0} G[\Gamma_t(V)] = \partial_t z(0, \psi_0)$$

Some more calculations by integration by parts lead to

$$\partial_t \dot{z}(0,\psi_0) = \int_\Gamma \left[\frac{\varepsilon_+}{2} |\nabla\psi_0^+|^2 - \frac{\varepsilon_-}{2} |\nabla\psi_0^-|^2 + B(\psi_0) \right] (V\cdot n)\, ds$$
$$- \int_\Gamma \varepsilon_+ (\nabla\psi_0^+ \cdot n)(V\cdot\nabla\psi_0^+)\, ds + \int_\Gamma \varepsilon_- (\nabla\psi_0^- \cdot n)(V\cdot\nabla\psi_0^-)\, ds$$

Since $\psi_0^+ = \psi_0^-$ along Γ.

$$\nabla(\psi_0^+ - \psi_0^-) = (\nabla\psi_0^+ \cdot n - \nabla\psi_0^- \cdot n)\, n \quad \text{on } \Gamma$$

Also, $\varepsilon_+ \nabla\psi_0^+ \cdot n = \varepsilon_- \nabla\psi_0^- \cdot n = \varepsilon_\Gamma \nabla\psi_0 \cdot n$ on Γ

Use these, we obtain the final result.

Notes: $\nabla\psi_0^\pm = (\nabla\psi_0^\pm \cdot n)n + (I - n\otimes n)\nabla\psi_0^\pm$ on Γ.

$$\underbrace{(I - n\otimes n)\nabla\psi_0^+}_{} = \overbrace{(I - n\otimes n)\nabla\psi_0^-}^{} \quad \text{on } \Gamma.$$

tangential part of $\nabla\psi_0$ on Γ. $\Box$