Longo: Math 10B - Winter 2017
Lecture Notes

Date: February 3, 2017

Section:
- §7.2

Topics Covered:
- Integration by parts

§7.2: Integration by Parts (The anti-product rule)

Recall: (Product Rule) Let u, v be diff'ble.

$$\boxed{(u\cdot v)' = u'v + uv'}$$

Let's see how this can help us integrate.

$$(u\cdot v)' = u'\cdot v + uv'$$

$$\Rightarrow \quad (u\cdot v)' - u'\cdot v = uv'$$

If we integrate with respect to x on both sides, we get:

$$\int (u\cdot v)'\,dx - \int u'\cdot v\,dx = \int uv'\,dx$$

$$\Rightarrow \quad \boxed{\int uv'\,dx = uv - \int vu'\,dx} \quad (*)$$

Q? Why does this help?

Ans: If we are careful, the integral on the right will be easier to calculate.

Example: $\int t\ln(t)\,dt$

Idea: We don't know an antiderivative of $\ln(t)$ right now, but we do know its derivative. So if we let $u = \ln(t)$, then maybe the integral on the right will be easy.

Let $u = \ln(t)$ $\quad v' = t$

$u' = \frac{1}{t}$ $\quad v = \frac{t^2}{2}$ $\quad$ (Note: you can ignore the +C here)

By (*), $\int t\ln(t)\,dt = \frac{t^2}{2}\ln(t) - \int \frac{t^2}{2t}\,dt$

$$= \frac{t^2\ln(t)}{2} - \frac{1}{2}\int t\,dt$$

$$= \boxed{\frac{t^2\ln(t)}{2} - \frac{t^2}{4} + C.}$$

Let's check by taking derivatives:

$$\frac{d}{dt}\left(\frac{1}{2}\left(t^2\ln(t)\right) - \frac{t^2}{4}\right) = t\ln(t) + \frac{t^2}{2t} - \frac{2t}{4}$$

$$= t\ln(t) + \frac{t}{2} - \frac{t}{2}$$

$$= t\ln(t) \quad \checkmark.$$

Remark: (1) If you pick u to be something that has a simple derivative and v' to be something with a simple antiderivative, integration by parts works.

(2) When you are picking 'u', remember: $\boxed{\text{LIATE}}$. This is the priority for picking u.

L — logarithmic

I — inverse trigonometric

A — algebraic (polynomial)

T — trigonometric

E — exponential

Simpler derivatives ↕ less simple derivatives.

$\boxed{\text{Q?}}$ How can you recognize when to use integration by parts?

$\boxed{\text{Ans:}}$ Use integration by parts when the integrand is a product. You can remember this because the formula comes from the product rule.

Examples: 1. $\int x e^{-x}$

Here, we have the polynomial 'x' times the exponential e^{-x}. According to LIATE, we should let

$$u = x \qquad v' = e^{-x}$$
$$u' = 1 \qquad v = -e^{-x}$$

Then by (*),

$$\int x e^{-x}\,dx = -xe^{-x} - \int (1)(-e^{-x})\,dx$$
$$= -xe^{-x} + \int e^{-x}\,dx$$
$$\boxed{= -xe^{-x} - e^{-x} + C}$$

2. $\int x^2 \sin(x)\,dx.$

Here, we have the polynomial 'x^2' times the trig. fcn $\sin(x)$. According to LIATE, we should let

$$u = x^2 \qquad v' = \sin(x). \quad \text{Then}$$
$$u' = 2x \qquad v = -\cos(x)$$

By (*),

$$\int x^2 \sin(x)\,dx = -x^2\cos(x) - \int 2x(-\cos(x))\,dx$$
$$= -x^2\cos(x) + 2\int x\cos(x)\,dx$$

Problem: We are still left with $\int x\cos(x)\,dx$, which still isn't obvious. However, the 'x^2' in the original integral became 'x' in the new integral. If we use integration by parts again, we might be able to complete it.

Subproblem: Calculate $\int x\cos(x)\,dx.$

$$u = x \qquad v' = \cos(x)$$
$$u' = 1 \qquad v = \sin(x)$$

By (*) $\int x\cos(x)\,dx = x\sin(x) - \int 1\cdot\sin(x)\,dx = x\sin(x) + \cos(x)$

All together: $\boxed{\int x^2\sin(x)dx = -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C}$

(3) $\int_0^1 \tan^{-1}(x)\,dx$ (Reminder: $\tan^{-1}(x) = \arctan(x)$).

Let's first find the indefinite integral $\int \tan^{-1}(x)dx$

This one is tricky. We don't know an antiderivative of $f(x) = \tan^{-1}(x)$, but we do know $\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$.

Let's rewrite the integral: $\int \tan^{-1}(x)dx = \int 1\cdot\tan^{-1}(x)\,dx$.

Now we have the constant fcn 1 times $\tan^{-1}(x)$. According to LIATE, we should let

$$u = \tan^{-1}(x) \qquad v' = 1. \qquad \text{Then,}$$
$$u' = \frac{1}{1+x^2} \qquad v = x$$

By (*): $\int \tan^{-1}(x)dx = x\tan^{-1}(x) - \int \frac{x}{1+x^2}\,dx$

Now for $\int \frac{x}{1+x^2}dx$, we can use substitution:

$$w = 1+x^2 \Rightarrow \frac{dw}{dx} = 2x \Rightarrow \frac{1}{2}dw = x\,dx$$

So $\int \frac{x}{1+x^2}dx = \int \frac{1}{w}\left(\frac{1}{2}\right)dw = \frac{1}{2}\int \frac{1}{w}dw = \frac{1}{2}\ln|w| + C$
$= \frac{1}{2}\ln|1+x^2|$

All together: $\int \tan^{-1}(x)dx = x\tan^{-1}(x) - \frac{1}{2}\ln|1+x^2| + C$, so

$$\int_0^1 \tan^{-1}(x)dx = \left(x\tan^{-1}(x) - \frac{1}{2}\ln|1+x^2|\right)\Big|_0^1$$
$$= \left(\tan^{-1}(1) - \frac{1}{2}\ln(2)\right) - \left(0\cdot\cancel{\tan^{-1}(0)} - \frac{1}{2}\cancel{\ln(1)}\right)$$
$$= \tan^{-1}(1) - \frac{1}{2}\ln(2)$$
$$\boxed{= \frac{\pi}{4} - \frac{1}{2}\ln(2)}$$

(4) $\int e^x\sin(x)dx$.

Here, we have a product of an exponential 'e^x' and a trig. fcn $\sin(x)$. According to LIATE we should let $u = \sin(x)$ $v' = e^x$.

Notice, trig and exp. are both low priority.

Maybe we should expect something weird to happen...

$$u = \sin(x) \qquad v' = e^x$$
$$u' = \cos(x) \qquad v = e^x$$

By $(*)$ $\int e^x \sin(x)\,dx = e^x \sin(x) - \int e^x \cos(x)\,dx.$

... Uhh. $\int e^x \cos(x)\,dx$ doesn't look much easier, let's try again.

$$u = \cos(x) \qquad v' = e^x$$
$$u' = -\sin(x) \qquad v = e^x$$

By $(*)$
$$\int e^x \cos(x)\,dx = e^x \cos(x) - \int e^x(-\sin(x))\,dx$$
$$= e^x \cos(x) + \int e^x \sin(x)\,dx.$$

All together, $\int e^x \sin(x)\,dx = e^x \sin(x) - \int e^x \cos(x)\,dx$

$(*)$ $\int e^x \sin(x)\,dx = e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x)\,dx$

The original popped out! WTF?

$(*) \Rightarrow \quad 2\int e^x \sin(x)\,dx = e^x \sin(x) - e^x \cos(x)$

$$\Rightarrow \quad \boxed{\int e^x \sin(x)\,dx = \frac{e^x \sin(x) - e^x \cos(x)}{2} + C}$$