Longo: Math 10B - Winter 2017
Lecture Notes

Date: February 6, 2017

Section:
- ☐ §7.3
- ☐ §7.4

Topics Covered:
- ☐ Tables
- ☐ The method of partial fraction decomposition

§7.3: Tables: If you get comfortable using the methods of substitution and integration by parts, you'll notice that common patterns will emerge. In order to save time, we use these methods to create formula tables so that we don't have to start from scratch every time. You can find such a table on the back cover of your book.

Examples: ① $\int e^{5x}\cos(3x)\,dx$.

Sol: Your table has an entry: II.9: $\int e^{ax}\cos(bx)\,dx = \frac{1}{a^2+b^2}e^{ax}[a\cos(bx)+b\sin(bx)]+C$

Here $a=5$, $b=3$, so $\boxed{\int e^{5x}\cos(3x)\,dx = \left(\frac{1}{5^2+3^2}\right)e^{5x}[5\cos(3x)+3\sin(3x)]+C}$

Rmk: This formula comes from using IBP twice. See the example from the lecture notes on Feb. 3rd for details.

② $\int \frac{3}{x^2+4x+1}\,dx$

Sol: The denominator is an irreducible quadratic polynomial. I.e., the denominator cannot be factored. The most similar integral in the table is

V.24: $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

Rmk: You get this by factoring $\int \frac{1}{x^2+a^2}\,dx = \int \frac{1}{a^2\left(\left(\frac{x}{a}\right)^2+1\right)}\,dx$ and using substitution with $u=\frac{x}{a}$. Try it!

To put our integral in this form, we complete the square:

$$x^2+4x+8 = \left(x^2+4x+\left(\tfrac{4}{2}\right)^2\right) - \left(\tfrac{4}{2}\right)^2+8$$
$$= (x+2)^2+4$$

$\Rightarrow \int \frac{3}{(x+2)^2+4}\,dx = \int \frac{3}{x^2+4x+8}\,dx.$ Now substitute: $u = x+2$
$\Rightarrow \frac{du}{dx}=1$
$\Rightarrow du = dx$

$$\int \frac{3}{(x+2)^2+4}\,dx = 3\int \frac{1}{u^2+4}\,du$$
$$= 3\int \frac{1}{u^2+2^2}\,du$$
Table. $$= 3\left(\tfrac{1}{2}\tan^{-1}\left(\tfrac{u}{2}\right)\right)+C$$
$$= \boxed{3\left(\tfrac{1}{2}\tan^{-1}\left(\tfrac{x+2}{2}\right)\right)+C}$$

You will get more practice using tables in the homework.

§7.4 (Part 1): The method of partial fraction decomposition:

Motivating example: $\int \frac{x+1}{x^2+2x}\,dx.$

The idea is to decompose the rational fcn $\frac{x+1}{x^2+2x}$ into a sum of simpler rational fcns. We start by factoring the denominator. $x^2+2x = x(x+2)$. Let's try to write

$\frac{x+1}{x^2+2x}$ as $\frac{A}{x} + \frac{B}{x+2}$ for some constants A, B.

If $\frac{x+1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$ we "clear denominators":

$$\Rightarrow \cancel{x(x+2)}\left(\frac{x+1}{\cancel{x(x+2)}}\right) = (\cancel{x}(x+2))\left(\frac{A}{\cancel{x}}\right) + \left(\frac{B}{\cancel{x+2}}\right)(x\cancel{(x+2)})$$

$\Rightarrow x+1 = A(x+2) + Bx$ "recollect common terms"

$x+1 = (A+B)x + 2A$. Now by comparing coefficients for the constant term, and in front of the x, we get a system of equations:

$$\left.\begin{aligned} A+B &= 1 \\ 2A &= 1 \end{aligned}\right\} \Rightarrow \boxed{A = \tfrac{1}{2}} \Rightarrow \quad \tfrac{1}{2} + B = 1 \Rightarrow \boxed{B = \tfrac{1}{2}}$$

So
$$\int \frac{x+1}{x^2+2x}\,dx = \int \frac{\frac{1}{2}}{x}\,dx + \int \frac{\frac{1}{2}}{x+2}\,dx$$
$$= \frac{1}{2}\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{1}{x+2}\,dx$$
$$= \frac{1}{2}\ln|x| + \frac{1}{2}\ln|x+2| + C$$

Rmk: The integral $\int \frac{1}{x-2}\,dx$ requires the substitution $u = x+2 \Rightarrow du = dx$. I leave the details to the reader.

Example ②: Irreducible factors.
$$\int \frac{x^2}{(x^2+1)(x-2)}\,dx.$$

Note that the factor in the denominator, x^2+1, cannot be factored further. We should therefore try to decompose

$$\frac{x^2}{(x^2+1)(x-2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-2}$$

$\boxed{\text{Rmk:}}$ We need a linear numerator: $Ax+B$ because the denominator is quadratic.

Again, clear denominators:

$$(x^2+1)(x-2)\left(\frac{x^2}{(x^2+1)(x-2)}\right) = (\cancel{x^2+1})(x-2)\left(\frac{Ax+B}{\cancel{x^2+1}}\right) + (x^2+1)(\cancel{x-2})\left(\frac{C}{\cancel{x-2}}\right)$$

$$\Rightarrow \quad x^2 = (x-2)(Ax+B) + (x^2+1)C \qquad \text{recollect terms} \Rightarrow$$
$$x^2 = (A+C)x^2 + (B-2A)x + (C-2B)$$

By comparing coefficients we get the system of eqns:

$$\left.\begin{aligned} A+C &= 1 \\ B-2A &= 0 \\ C-2B &= 0 \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} B &= 2A \\ C &= 2B \end{aligned}} \Rightarrow \boxed{\begin{aligned} B &= 2A \\ C &= 4A \end{aligned}} \quad \text{plug into } C+A=1.$$

$$4A + A = 1 \Rightarrow \boxed{A = \tfrac{1}{5}} \Rightarrow \boxed{B = \tfrac{2}{5},} \quad \boxed{C = \tfrac{4}{5}}$$

So
$$\int \frac{x^2}{(x^2+1)(x-2)}\,dx = \int \frac{\frac{1}{5}x + \frac{2}{5}}{x^2+1}\,dx + \int \frac{\frac{4}{5}}{x-2}\,dx$$
$$= \boxed{\frac{1}{10}\ln(x^2+1) + \frac{2}{5}\tan^{-1}(x) + \frac{4}{5}\ln|x-2| + C}$$

Table V. 25. (we will see how to do this next class)

Example ③: (Repeated roots) $\int \frac{x+1}{(x-1)^2(x+2)}\,dx$

Sol: Since $(x-1)^2$ is a repeated root we use the form

$$\frac{x+1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}.$$ Clear denominators again.

$$\begin{aligned} x+1 &= (x-1)(x+2)A + B(x+2) + C(x-1)^2 \\ &= (x^2+x-2)A + Bx + 2B + C(x^2-2x+1) \\ &= (A+C)x^2 + (A+B-2C)x + (-2A+2B+C) \end{aligned}$$

$$\Rightarrow \left.\begin{aligned} A+C &= 0 \\ A+B-2C &= 1 \\ -2A+2B+C &= +1 \end{aligned}\right\} \quad A = -C \quad \Rightarrow \begin{aligned} -C+B-2C &= 1 \\ 2C+2B+C &= -1 \end{aligned}$$

$$\rightarrow \left.\begin{aligned} -3C+B &= 1 \\ 3C+2B &= +1 \end{aligned}\right\} \Rightarrow 3B = 2 \quad \boxed{B = \tfrac{2}{3}}$$

$\Rightarrow \quad -3C + \frac{2}{3} = 1 \Rightarrow -3C = \frac{1}{3} \Rightarrow \boxed{C = -\frac{1}{9}} \Rightarrow \boxed{A = \frac{1}{9}}$

So $\displaystyle\int \frac{x+1}{(x-1)^2(x+2)}dx = \int \frac{\frac{1}{9}}{(x-1)}dx - \int \frac{\frac{1}{9}}{(x-1)^2}dx + \frac{2}{3}\int \frac{1}{x+2}dx$

$\displaystyle = \frac{1}{9}\ln|x-1| + \underline{\frac{1}{9}(x-1)^{-1}} + \frac{2}{3}\ln|x+2| + C$

Use the substitution $u = x-1$ here.