Longo: Math 10B - Winter 2017
Lecture Notes

Date: February 8, 2017

Section:

- [ ] §7.4

Topics Covered:

- [ ] Partial fraction decomposition and trig substitution

From last time:

We saw that if you have a rational fcn, you can decompose it as a sum of partial fractions in order to integrate. For example: $\frac{1}{(x+1)(x-1)} = \frac{1/2}{x+1} + \frac{1/2}{x-1}$

Q? Here we used the "form" $\frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$ for constants A, B. In general, what should your decomposition look like?

Ans: (1) If deg. of numerator $\geq$ deg of denom.
Use polynomial long division first, and use partial fractions on the remainder.

(2) Factor the denominator as much as possible.

(3) For every factor of the form $x-r$ (with no exponent) you need a term of the form

$$\boxed{\frac{A}{x-r}}$$

(4) If the denom has an irreducible quadratic, $q(x)$, you need a term of the form

$$\boxed{\frac{Ax+B}{q(x)}}$$

(5) If the denom has a repeated root, i.e., $(x-r)^n$ you need to include

$$\boxed{\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \dots + \frac{A_n}{(x-r)^n}}$$

Example: (Shortcut that works exceptionally well when you have distinct roots)

$$\int \frac{x-5}{(x+5)(x-2)}\,dx. \qquad \frac{x-5}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2}$$

$$x-5 = A(x-2) + B(x+4)$$

Now plug in $x=-4$ and $x=-2$.

$\underline{x=-4:}$ $$-4-5 = A(-4+2) + B(-4+4)$$
$$\Rightarrow \quad -9 = 2A$$
$$\Rightarrow \quad \boxed{A = \frac{-9}{2}}$$

$\underline{x=2:}$ $$2-5 = A\underbrace{(x-2)}_{0} + B(2-4)$$
$$\Rightarrow \quad -3 = -2B$$
$$\Rightarrow \quad \boxed{B = \frac{3}{2}}$$

$$\int \frac{x-5}{(x+4)(x-2)}dx = \int \frac{-\frac{9}{2}}{x+4}dx + \int \frac{\frac{3}{2}}{x-2}dx$$
$$= \boxed{-\frac{9}{2}\ln|x+4| + \frac{3}{2}\ln|x-2| + C}$$

Remark: ① In the original function, $x=-4$, $x=2$ were not in the domain, so it doesn't make sense to plug in these numbers. Nevertheless, since polynomials are continuous, equality holds after you clear denominators.

② This method also works when you have repeated roots or irreducible factors in the denom, but its a bit harder because you don't have vanishing.

Trig. Substitution:

We now discuss a special type of substition that works particularly well when we see things that look like the Pythagorean identities.

Idea: Say we want $\int \frac{2x+3}{x^2+9}\,dx$.

This kind of thing happens often when doing partial fractions for example. The denominator is not factorable, so we cannot decompose into partial fractions. Let's separate the terms in the numerator:

$$\int \frac{2x+3}{x^2+9}\,dx = \int \frac{2x}{x^2+9}\,dx + \int \frac{3}{x^2+9}\,dx$$

For $\int \frac{2x}{x^2+9}\,dx$, let $u = x^2+9$, $\Rightarrow du = 2x\,dx$

$$\Rightarrow \int \frac{2x}{x^2+9}\,dx = \int \frac{1}{u}\,du = \ln|u| + C$$
$$= \ln(x^2+9) + C$$

For $\int \frac{3}{x^2+9}\,dx$, we notice the denominator is $x^2+(3)^2$.
The trick is to use the Pythagorean Identity:
$\tan^2\theta + 1 = \frac{1}{\cos^2\theta}$. This identity implies $(3\tan\theta)^2 + 3^2 = \frac{9}{\cos^2\theta}$.

So let $x = 3\tan(\theta)$. Then $\frac{dx}{d\theta} = 3\left(\frac{1}{\cos^2\theta}\right) \Rightarrow dx = 3\left(\frac{1}{\cos^2\theta}\right)d\theta$

Then
$$\int \frac{3}{x^2+9}\,dx = \int \frac{3}{(3\tan\theta)^2+9} \cdot \underbrace{\left(\frac{3}{\cos^2\theta}\right)d\theta}_{dx}$$
$$= \int \left(\frac{9}{9\tan^2\theta+9}\right)\left(\frac{1}{\cos^2\theta}\right)d\theta$$
$$= \int \left(\frac{9}{9(\tan^2\theta+1)}\right)\left(\frac{1}{\cos^2(\theta)}\right)d\theta$$
$$= \int \left(\frac{1}{\left(\frac{1}{\cos^2\theta}\right)}\right)\cdot\left(\frac{1}{\cos^2\theta}\right)d\theta$$
$$= \int \frac{\cos^2\theta}{\cos^2\theta}\,d\theta$$
$$= \int 1\,d\theta$$
$$= \theta + C.$$

Now let's rewrite θ in terms of x. Since
$$x = 3\tan\theta$$

we have $\frac{x}{3} = \tan\theta$, and so $\theta = \tan^{-1}\left(\frac{x}{3}\right)$.

Therefore, $\boxed{\int \frac{3}{x^2+9}\,dx = \tan^{-1}\left(\frac{x}{3}\right) + C}$

Summary: To "simplify" x^2+a^2, we use the substitution

$$x = a\tan\theta \Rightarrow (a\tan\theta)^2 + a^2 = a^2\tan^2\theta + a^2$$
$$= a^2(\tan^2\theta + 1)$$
$$= a^2$$

and $dx = \frac{a}{\cos^2\theta}\,d\theta$.

On the other hand, what if we see $x^2 - a^2$?
We have the Pythagorean Identity: $\sin^2\theta - 1 = \cos^2\theta$.
So maybe we should try to substitute $x = a\sin\theta$.

Example: $\int_{-3}^{3} \sqrt{9-x^2}\,dx$. Since there is a $(-)$ sign instead of $(+)$, we should use a Sine sub. Let's first ignore the bounds, and calculate the indefinite integral.

$$\int \sqrt{9-x^2}\,dx = \int \sqrt{(3)^2 - x^2}\,dx. \quad \text{Let } x = 3\sin\theta.$$

then $\frac{dx}{d\theta} = 3\cos\theta \Rightarrow dx = 3\cos\theta\,d\theta$

Substituting, we get:

$$\int \sqrt{9 - \underbrace{(3\sin\theta)}_{x}{}^2}\;\underbrace{(3\cos\theta\,d\theta)}_{dx}$$
$$= \int \left(\sqrt{9 - 9\sin^2\theta}\right)(3\cos\theta)\,d\theta$$
$$= \int \sqrt{9(1-\sin^2\theta)}\,(3\cos\theta)\,d\theta$$
$$= \int \left(\sqrt{9\cos^2\theta}\right)(3\cos\theta)\,d\theta$$
$$= \int (3\cos\theta)(3\cos\theta)\,d\theta$$
$$= 9\int \cos^2\theta\,d\theta$$

Now we can use the Power Reducing identity

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta).$$

So $9\int \cos^2\theta\, d\theta = 9\int \frac{1}{2} + \frac{1}{2}\cos 2\theta \, d\theta$
$= 9\left(\frac{\theta}{2} + \frac{1}{2}\sin(2\theta)\right) + C.$

To deal with the bounds $x=3$, $x=-3$. Notice when $x=3$, we have $3 = 3\sin\theta \Rightarrow 1 = \sin\theta$
$\Rightarrow \theta = \frac{\pi}{2}.$

when $x=-3$, $-3 = 3\sin\theta \Rightarrow \sin\theta = -1$
$\Rightarrow \theta = \sin^{-1}(-1) = -\frac{\pi}{2}.$

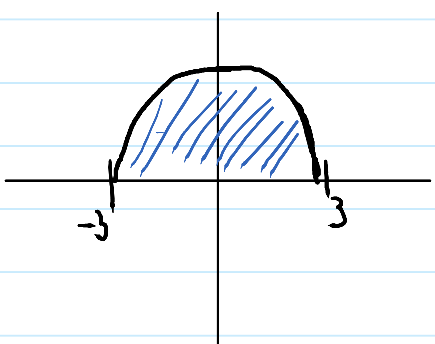
All together: $\int_{-3}^{3} \sqrt{9-x^2}\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 9\cos^2\theta\, d\theta$

$$= 9\left(\frac{1}{2}\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 9\left(\frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{2}\overset{0}{\cancel{\sin}(\pi)}\right) - 9\left(\frac{1}{2}\left(\frac{-\pi}{2}\right) + \frac{1}{2}(\overset{0}{\cancel{\sin}(-\pi)})\right)$$

$$= 9\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \boxed{9\left(\frac{\pi}{2}\right)}.$$

$\boxed{Q?}$ What did we just calculate?



$$y = \sqrt{9-x^2} \Rightarrow y^2 = 9 - x^2$$
$$\Rightarrow y^2 + x^2 = 9$$

The graph is the upper hemisphere of the circle w/ radius 3.

So $\int_{-3}^{3} \sqrt{9-x^2}\, dx = \frac{1}{2}(\text{Area of Circle})$
$= \frac{1}{2}(9\pi).$

Example: $\int \frac{3}{\sqrt{16-25x^2}}\,dx$. This doesn't quite look like a^2-x^2, so let's manipulate it first:

$$16-25x^2 = 25\left(\tfrac{16}{25}-x^2\right).$$
$$= 25\left(\left(\tfrac{4}{5}\right)^2-x^2\right).$$

So let's make the substitution: $x = \frac{4}{5}\sin(\theta) \Rightarrow dx = \frac{4}{5}\cos\theta\,d\theta$

$$\int \frac{3}{\sqrt{16-25x^2}}\,dx = \int\left(\frac{3}{\sqrt{16-25\underbrace{\left(\tfrac{4}{5}\sin\theta\right)}_{x}^2}}\right)\underbrace{\left(\tfrac{4}{5}\cos\theta\,d\theta\right)}_{dx}$$

$$= \int \frac{3}{\sqrt{16-25\left(\frac{16}{25}\right)\sin^2\theta}}\left(\tfrac{4}{5}\cos\theta\right)d\theta$$

$$= \int \frac{3}{\sqrt{16-16\sin^2\theta}}\left(\tfrac{4}{5}\cos\theta\right)d\theta$$

$$= \int \frac{3}{\sqrt{16(1-\sin^2\theta)}}\left(\tfrac{4}{5}\cos\theta\right)d\theta$$

$$= \int \frac{3}{\sqrt{16\cos^2\theta}}\left(\tfrac{4}{5}\cos\theta\right)d\theta$$

$$= \int \frac{3}{4\cancel{\cos\theta}}\left(\tfrac{4}{5}\cancel{\cos\theta}\right)d\theta$$

$$= \int \frac{3}{5}\,d\theta$$

$$= \frac{3}{5}\theta + C.$$

Now since $x = \frac{4}{5}\cos\theta \Rightarrow \frac{5x}{4} = \cos\theta$

$$\Rightarrow \cos^{-1}\left(\tfrac{5x}{4}\right) = \theta$$

So $\boxed{\int \frac{3}{\sqrt{16-25x^2}}\,dx = \frac{3}{5}\cos^{-1}\left(\frac{5x}{4}\right) + C}$