Longo: Math 10B - Winter 2017
Lecture Notes

Date: February 13, 2017

Section:

- §7.6

Topics Covered:

- Improper integrals

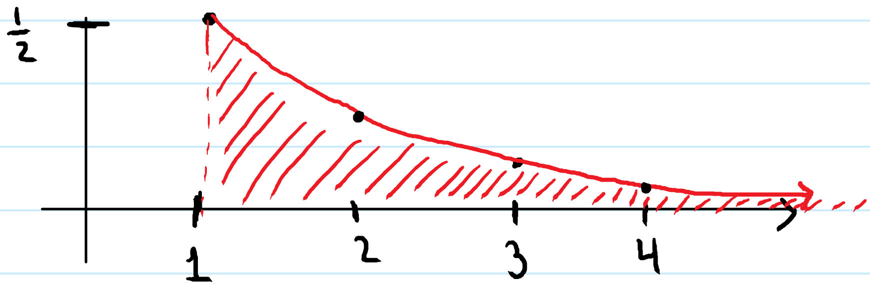
Related Homework Problems:

§7.6: Improper integrals:

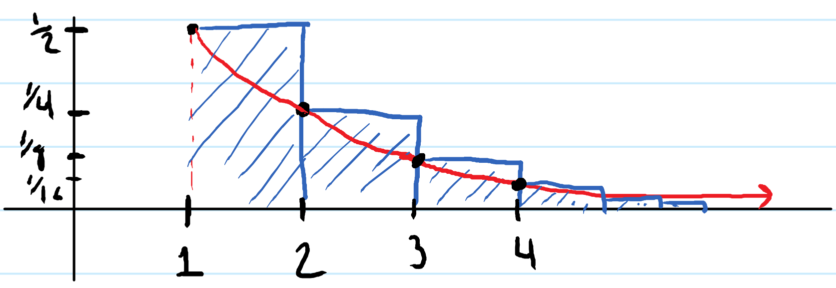
Idea: What happens if we let one (or both) of the bounds of integration be $\pm\infty$.

Example: $\int_1^\infty \left(\frac{1}{2}\right)^x dx$.

Our integration intuition tells us that this represents the area under the curve of $f(x) = \left(\frac{1}{2}\right)^x$ from 1 to ∞.



Point of confusion: Since $x \to \infty$, the region between the graph and the x-axis also "goes off to ∞" (we say the region is unbounded). A priori, we shouldn't expect the total area to be a finite number. To gain understanding, let's look at an "infinite" left-hand sum, which is an upper bound since $f(x) = \left(\frac{1}{2}\right)^x$ is decreasing.

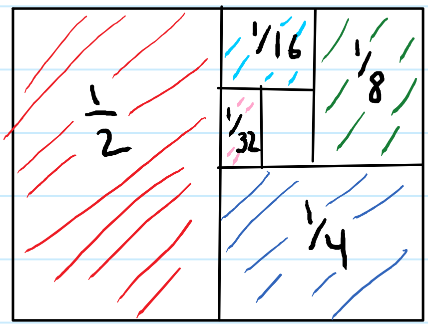


Using $\Delta x = 1$, the first box has area $\frac{1}{2}$, the second has area $\frac{1}{4}$, the third has area $\frac{1}{8}$, and so on.

As an upper bound, we can say

$$\int_1^\infty \left(\tfrac{1}{2}\right)^x dx \le \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \dots$$

The right side of the inequality is an infinite sum, (more on this later). I claim the infinite sum is equal to 1. To visualize this, start with a square of area 1. We chop the square in half and fill in one of the halves. Chop the unshaded rectangle in half and fill in one of the pieces. So far, the shaded area is $\tfrac{1}{2} + \tfrac{1}{4}$. If we continue this process forever, the shaded area will be 1 (the entire square). On the other hand, the shaded area is $\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \dots$.

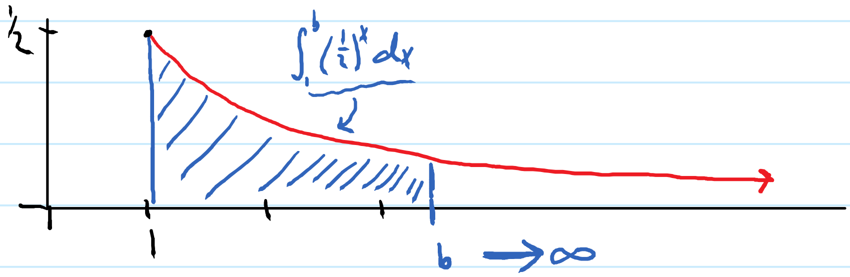


So $\boxed{\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \dots = 1}$.

Now we have: $\int_1^\infty \left(\tfrac{1}{2}\right)^x dx \le \tfrac{1}{2} + \tfrac{1}{4} + \dots = 1$. So we have reason to believe the total area is finite even though the region is unbounded. Intuitively, since the "infinite end" of the region is so thin that the total area is finite.

How can we actually calculate?

Fix some number, b and calculate $\int_1^b \left(\tfrac{1}{2}\right)^x dx$. $\int_1^b \left(\tfrac{1}{2}\right)^x dx$ is an expression in terms of b that represents the area under the curve from 1 to b. Then let b go to ∞ to get the area from 1 to ∞.



$$\int_1^b \left(\tfrac{1}{2}\right)^x dx = \left(\frac{\left(\frac{1}{2}\right)^x}{\ln\left(\frac{1}{2}\right)} \right)\Bigg|_{x=1}^{x=b} = \underbrace{\frac{1}{\ln\left(\frac{1}{2}\right)}\left(\left(\tfrac{1}{2}\right)^b - \left(\tfrac{1}{2}\right)^1\right)}_{\text{Area from 1 to b.}}$$

$$\int_1^\infty \left(\tfrac{1}{2}\right)^x dx = \lim_{b\to\infty}\left(\int_1^b \left(\tfrac{1}{2}\right)^x dx\right) = \lim_{b\to\infty}\left(\frac{1}{\ln\left(\frac{1}{2}\right)}\left(\cancel{\left(\tfrac{1}{2}\right)^b}^{\,0} - \left(\tfrac{1}{2}\right)^1\right)\right)$$

$$= \frac{1}{\ln\left(\frac{1}{2}\right)}\left(-\tfrac{1}{2}\right)$$

$$\boxed{\approx .721}$$

calculator

Definitions: Suppose f is cont. on the entire number line:

The following are called improper integrals:

① $\displaystyle\int_c^\infty f(x)\,dx = \lim_{b\to\infty}\left(\int_c^b f(x)\,dx\right)$

② $\displaystyle\int_{-\infty}^c f(x)\,dx = \lim_{b\to-\infty}\left(\int_b^c f(x)\,dx\right)$

③ $\displaystyle\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$

If the limits exist.

where c is any real number.

Examples: ① $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \int_1^\infty x^{-2}\,dx$

$$= \lim_{b\to\infty}\left(\int_1^b x^{-2}\,dx\right)$$

$$= \lim_{b\to\infty}\left(-x^{-1}\Big|_{x=1}^{x=b}\right)$$

$$= \lim_{b\to\infty}\left(-b^{-1} - (-1)\right)$$

$$= \lim_{b\to\infty} \left(-\cancel{\frac{1}{b}}^{\,0} + 1 \right) = \boxed{1}$$

② $$\int_0^\infty e^{-6x}\,dx = \lim_{b\to\infty} \left(\int_0^b e^{-6x}\,dx \right)$$

$$= \lim_{b\to\infty} \left(\left(-\frac{1}{6} e^{-6x} \right) \Big|_0^b \right)$$

$$= \lim_{b\to\infty} \left(-\frac{1}{6} \cancel{e^{-6b}}^{\,0} - \left(-\frac{1}{6} e^0 \right) \right)$$

$$= \boxed{\frac{1}{6}}$$

③ $$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty} \left(\int_1^b x^{-1/2}\,dx \right)$$

$$= \lim_{b\to\infty} \left(\left(\frac{x^{1/2}}{\frac{1}{2}} \right) \Big|_1^b \right) = \lim_{b\to\infty} \left(2\sqrt{x} \right) \Big|_1^b$$

$$= \lim_{b\to\infty} 2\sqrt{b} - 2\sqrt{1}$$

This limit does not exist ($2\sqrt{b} - 2 \to \infty$ as $b \to \infty$). In this case, we say the improper integral diverges.

We also obtain improper integrals when the integrand itself goes to $\pm\infty$.

If f is a fcn that goes to $\pm\infty$ at $x = c$, then

$$\int_a^c f(x)\,dx = \lim_{b\to c^-} \left(\int_a^b f(x)\,dx \right).$$

<u>Example:</u> $$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \int_0^1 x^{-1/2}\,dx.$$

Since $\frac{1}{\sqrt{x}} \to \infty$ as $x \to 0$, we need to use limits.

$$\int_0^1 x^{-\frac{1}{2}}dx = \lim_{b\to 0} \int_b^1 x^{-1/2}dx$$
$$= \lim_{b\to 0}\left(\left(\frac{x^{1/2}}{(\frac{1}{2})}\right)\Big|_b^1\right) = \lim_{b\to 0}\left((2\sqrt{x})\Big|_b^1\right)$$
$$= \lim_{b\to 0}\left(2\sqrt{1} - 2\sqrt{b}\right)$$
$$= \boxed{2}$$

Warning: Do not forget to write an improper integral as a limit!

Example: $\int_0^2 \frac{1}{(x-1)^2}dx$. Since $\frac{1}{(x-1)^2}$ goes to ∞ as $x \to 1$, we must write this integral as a sum of two improper integrals.

$$\int_0^2 \frac{1}{(x-1)^2}dx = \int_0^1 \frac{1}{(x-1)^2}dx + \int_1^2 \frac{1}{(x-1)^2}dx$$
$$= \lim_{b\to 1^-} \int_0^b \frac{1}{(x-1)^2}dx + \lim_{c\to 1^+} \int_c^2 \frac{1}{(x-1)^2}dx$$

Let's look at the left integral:

$$\lim_{b\to 1^-} \int_0^b \frac{1}{(x-1)^2}dx = \lim_{b\to 1^-}\left(-(x-1)^{-1}\Big|_0^b\right) = \lim_{b\to 1^-}\left(\frac{-1}{b-1} - \left(-\frac{1}{-1}\right)\right)$$

this limit does not exist, therefore

$\int_0^1 \frac{1}{(x-1)^2}\,dx$ diverges, which implies $\int_0^2 \frac{1}{(x-1)^2}\,dx$ diverges.