Longo: Math 10B - Winter 2016
Lecture Notes

Date: February 15, 2017

Section:

- §7.7

Topics Covered:

- Convergence test: Comparison of improper integrals

Related Homework Problems:

§7.7: Comparison tests for improper integrals.

$\boxed{Q?}$ Does $\int_1^\infty e^{-x^2}\,dx$ converge?

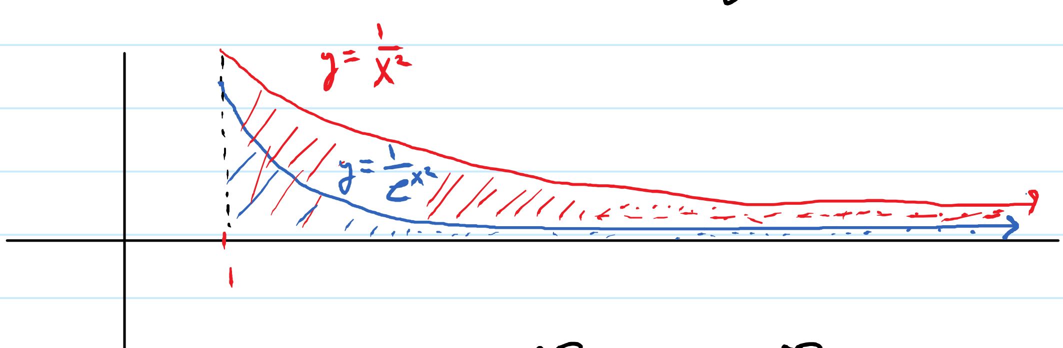
As we saw in section 6.4, we don't have a nice antiderivative for e^{-x^2}, so we cannot calculate $\lim_{b\to\infty}\left(\int_1^b e^{-x^2}dx\right)$ easily.

Therefore, we can't find the value of $\int_1^\infty e^{-x^2}dx$. Nevertheless, we can still figure out if the integral converges.

Notice that for $x \geq 1$, $e^{x^2} \geq x^2$ (to see this, show $e^{x^2} - x^2 \geq 0$ by using calc 1 techniques.). Therefore,

$$0 \leq \frac{1}{e^{x^2}} \leq \frac{1}{x^2} \quad \text{for } x \geq 1. \text{ I.e. the}$$

graph of $\frac{1}{x^2}$ is always above the graph of $\frac{1}{e^{x^2}}$



Using comparison tests: $0 \leq \int_1^\infty \frac{1}{e^{x^2}}dx \leq \int_1^\infty \frac{1}{x^2}\,dx.$

We saw last time $\int_1^\infty \frac{1}{x^2}dx = 1$. So

$$0 \leq \int_1^\infty \frac{1}{e^{x^2}}dx \leq 1.$$

Therefore $\int_1^\infty \frac{1}{e^{x^2}}dx$ does not go to infinity, so it must converge.

Rmk: Since $\frac{1}{e^{x^2}}$ is always positive, $\int_1^b \frac{1}{e^{x^2}}dx$ increases as b increases. Since $\int_1^b \frac{1}{e^{x^2}}dx$ is increasing and bounded above by 1, it must converge to a number.

Let's formalize this process:

Thm: Let f, g be positive fcns:

① If $f(x) \le g(x)$, $\left(\Rightarrow \int_a^b f(x)dx \le \int_a^b g(x)dx\right.$ where a or b could be $\pm\infty$ $\left.\right)$ and if $\int_a^\infty g(x)dx$ converges then $\int_a^\infty f(x)dx$ converges.

② If $g(x) \le f(x)$, so that $\int_a^\infty g(x)dx \le \int_a^\infty f(x)dx$, then if $\int_a^\infty g(x)$ diverges then $\int_a^\infty f(x)dx$ diverges

To use this thm, we should establish a bank of "simple" improper integrals that converge.

Last time we saw: $\int_1^\infty \frac{1}{x^2}dx$ converges, but $\int_1^\infty \frac{1}{\sqrt{x}}dx$ does not.

Q? For which positive numbers, p, does $\int_1^\infty \frac{1}{x^p}dx$ converge?

Ans! We must make a special case for $p=1$ since the anti-derivative of $\frac{1}{x}$ does not follow the pattern:

$p=1$:
$$\int_1^\infty \frac{1}{x}dx = \lim_{b\to\infty}\left(\int_1^b \frac{1}{x}dx\right) = \lim_{b\to\infty}\left(\ln|x|\Big|_{x=1}^{x=b}\right)$$
$$= \lim_{b\to\infty}\left(\ln b - \ln(1)\right)$$
$$= \lim_{b\to\infty} \ln b$$

$\ln b \to \infty$ as $x \to \infty$ so $\int_1^\infty \frac{1}{x}\,dx$ diverges.

If $p \neq 1$: $\int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty x^{-p}\,dx = \lim_{b\to\infty}\left(\int_1^b x^{-p}\,dx\right) = \lim_{b\to\infty}\left(\frac{x^{-p+1}}{-p+1}\Big|_1^b\right)$

$$= \boxed{\lim_{b\to\infty} \frac{b^{-p+1}}{-p+1} - \left(\frac{1^{-p+1}}{-p+1}\right)}$$

If $p>1$, then $-p+1<0$ so $\lim_{b\to\infty} \frac{b^{-p+1}}{-p+1} - \left(\frac{1}{-p+1}\right)$

$$= \lim_{b\to\infty} \frac{1}{(-p+1)\,b^{p-1}} - \left(\frac{1}{-p+1}\right)$$

$$= -\frac{1}{-p+1}$$

So if $p>1$, $\int_1^\infty \frac{1}{x^p}\,dx$ converges.

If $p<1$, $-p+1>0$, so $\lim_{b\to\infty}\left(\frac{b^{-p+1}}{-p+1} - \left(\frac{1}{-p+1}\right)\right)$

DNE since $b^{-p+1} \to \infty$ as $b \to \infty$.

So if $p \leq 1$, $\int_1^\infty \frac{1}{x^p}\,dx$ diverges.

Theorem: ① If $p>1$, $\int_1^\infty \frac{1}{x^p}\,dx$ converges

② If $p \leq 1$, $\int_1^\infty \frac{1}{x^p}\,dx$ diverges.

③ (from the initial example)

if $a>0$, $\int_1^\infty e^{-ax}\,dx$ converges.

Intuitively, this means if $p>1$, then $\frac{1}{x^p}$ shrinks fast enough so that the area under the graph is finite.

Examples: Decide if the following improper integrals converge:

① $\int_2^\infty \frac{1}{\sqrt{x^4+3x^3-1}}\,dx$.

Idea: Compare the integrand with a fcn of the form $\frac{1}{x^p}$ in order to decide convergence.

Note that as $x \to \infty$, the dominant term in the denominator is x^4. I.e. if x is huge,
$x^4 + 3x^3 - 1 \approx x^4$.

Therefore, $\sqrt{x^4+3x^3-1} \approx \sqrt{x^4} = x^2$, which means

$$\boxed{\frac{1}{\sqrt{x^4+3x^3-1}} \approx \frac{1}{x^2}}.$$

Since $2>1$, our guess is that $\int_2^\infty \frac{1}{\sqrt{x^4+3x^3-1}}\,dx$ converges. Let's use convergence test to prove it!

If $x \geq 2$, $3x^3 - 1 \geq 0$

$$\Rightarrow \quad x^4+3x^3-1 \geq x^4$$

$$\Rightarrow \quad \sqrt{x^4+3x^3-1} \geq \sqrt{x^4} = x^2$$

$$\Rightarrow \quad \frac{1}{\sqrt{x^4+3x^3-1}} \leq \frac{1}{x^2}$$

$$\Rightarrow \quad \int_2^\infty \frac{1}{\sqrt{x^4+3x^3-1}}\,dx \leq \underbrace{\int_2^\infty \frac{1}{x^2}\,dx}_{\text{Converges by "p-test".}}$$

By the comparison test $\int_2^\infty \frac{1}{\sqrt{x^4+3x^3-1}}\,dx$ converges.

Rmk: I still don't know what the integral is, I just know it converges.

(2) $\int_{10}^\infty \frac{3\sin^2(x)+1}{(x^7-1)^{1/4}}\,dx$

Note that since $0 \leq \sin(x) \leq 1$, $1 \leq 3\sin^2(x)+1 \leq 4$. This means the numerator doesn't really matter. In the denominator, the dominant term is x^7. If x is large, $(x^7-1)^{1/4} \approx (x^7)^{1/4}$.

So $\frac{1}{(x^7-1)^{1/9}} \approx \frac{1}{x^{7/9}}$ since $7/9 < 1$, we guess that our improper integral diverges. By the comparison test, we want to show

$$\boxed{\frac{3\sin^2(x)+1}{(x^7-1)^{1/9}} \geq \frac{1}{x^{7/9}}}$$

For $x \geq 10$ $\quad x^7-1 \leq x^7$

$\Rightarrow \quad (x^7-1)^{1/9} \leq (x^7)^{1/9}$

$$\Rightarrow \quad \boxed{\frac{1}{(x^7-1)^{1/9}} \geq \frac{1}{x^{7/9}}}$$

Since $\quad 4 \geq 3\sin^2 x + 1 \geq 1$,

$$\frac{3\sin^2 x+1}{(x^7-1)^{1/9}} \geq \frac{1}{x^{7/9}}$$

$$\Rightarrow \quad \int_{10}^{\infty} \frac{3\sin^2 x+1}{(x^7-1)^{1/9}}\,dx \geq \underbrace{\int_{10}^{\infty} \frac{1}{x^{7/9}}\,dx}_{\text{diverges}}$$

By the comparison test, $\int_{10}^{\infty} \frac{3\sin^2 x+1}{(x^7-1)^{1/9}}\,dx$ also diverges.

③ $\int_1^{\infty} \frac{\ln(x)}{x^4}\,dx$. (possibly difficult)

For $x > 1$, $\quad \ln(x) \leq x$

$$\Rightarrow \quad \frac{\ln(x)}{x^4} \leq \frac{x}{x^4} = \frac{1}{x^3}$$

$$\Rightarrow \quad \int_1^{\infty} \frac{\ln(x)}{x^4}\,dx \leq \underbrace{\int_1^{\infty} \frac{1}{x^3}\,dx}_{\text{converges}}$$

By the comparison test, $\int_1^\infty \frac{\ln(x)}{x^4}\,dx$ converges.