Longo: Math 10B - Winter 2017
Lecture Notes

Date: February 17, 2017

Section:

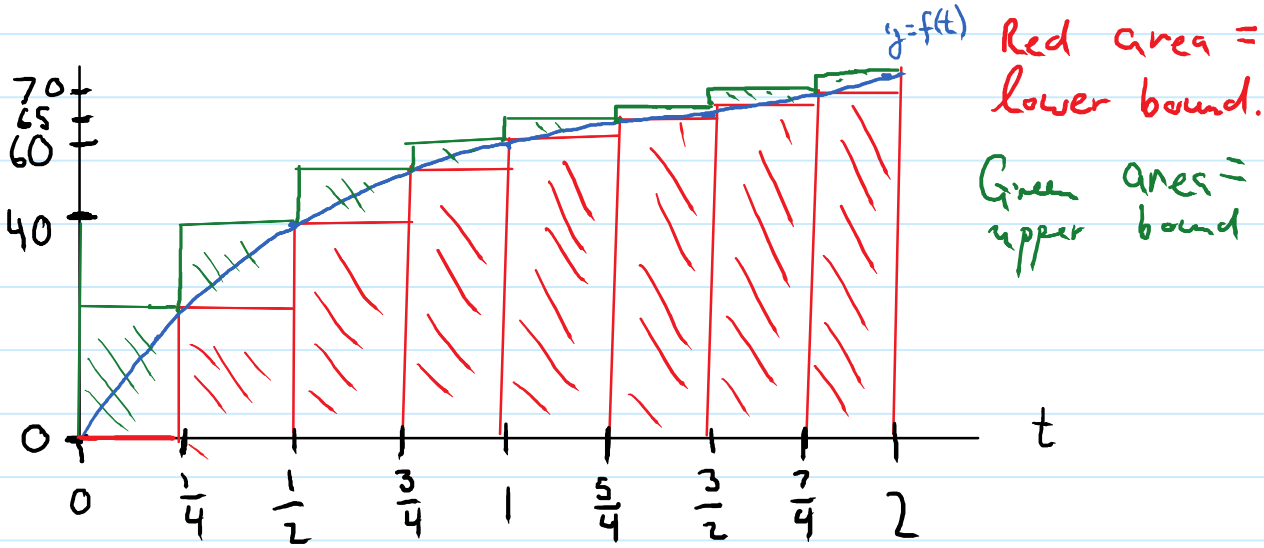
- [ ] §8.1

Topics Covered:

- [ ] Finding area and volume using slicing and integration

§8.1: Area and volume using cross sections and integration

Recall: When we wanted to calculate the area between the graph of a fcn f and the x-axis, we chopped the area up into rectangles in order to estimate:



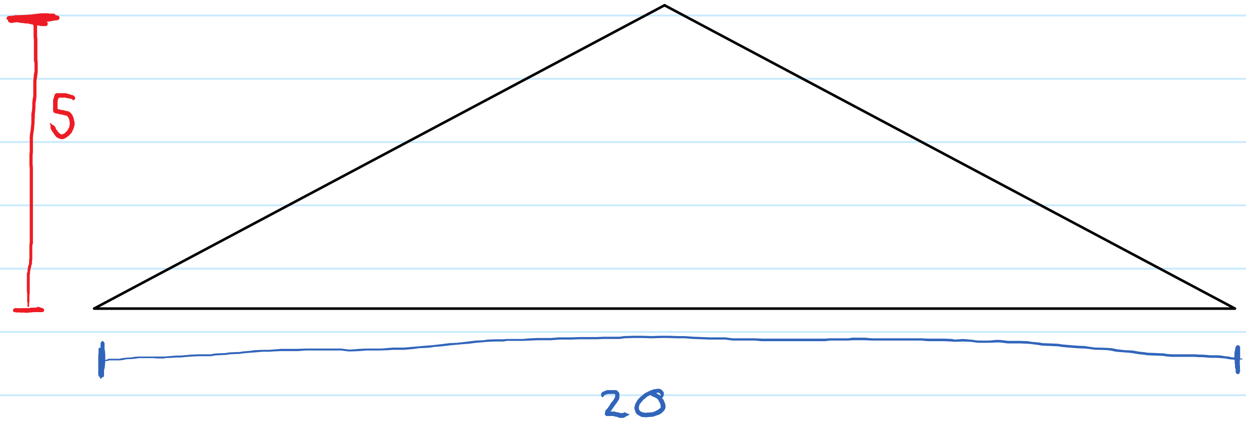
Then as the number of rectangles goes to infinity, we saw that the definite integral gave us the area.

We will now try to use a similar technique to compute more complicated areas and volumes.

Idea: To find the area or volume of a shape, calculate the area or volume of a horizontal or vertical slice (This will be a fcn of a single variable). Then integrate this fcn on an appropriate interval.

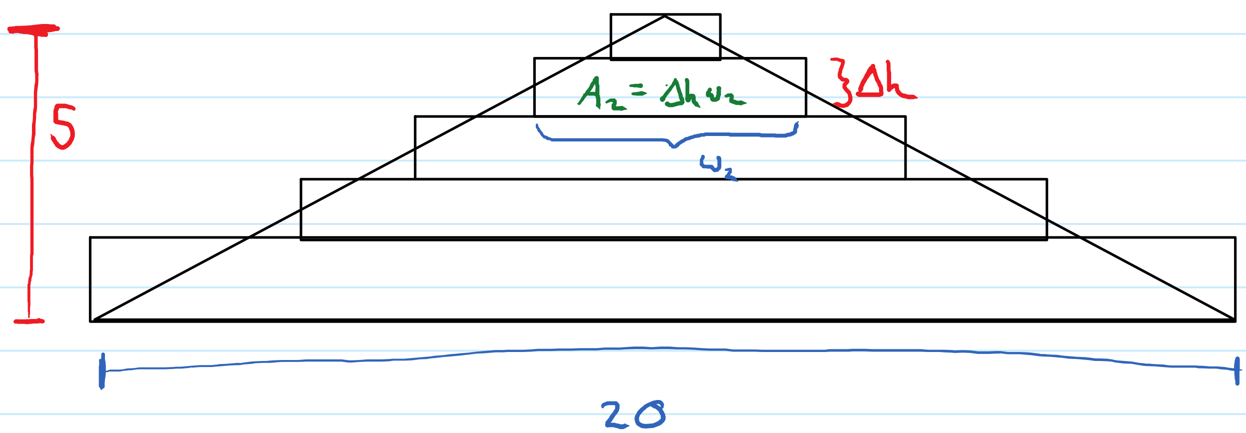
Let's start with a 2-dimensional example.

Example: Use horizontal slicing to calculate the area of the isosceles triangle!



Sol: Before we start, let's note that we already know the area is $\frac{1}{2}(20)(5) = 50$. OK, now let's overly complicate things in order to illustrate the technique.

If we cut the triangle into horizontal slices, we can approximate the area using rectangles.



In this picture, there are 5 rectangles. Suppose each rectangle has height Δh and the i^{th} rectangle has width w_i. Then the area of the i^{th} rectangle is $A_i = w_i \Delta h$. Then the area of the triangle is $\approx$ the sum of the areas of the rectangles: $A_1 + A_2 + \dots + A_5 = w_1 \Delta h + w_2 \Delta h + \dots + w_5 \Delta h$

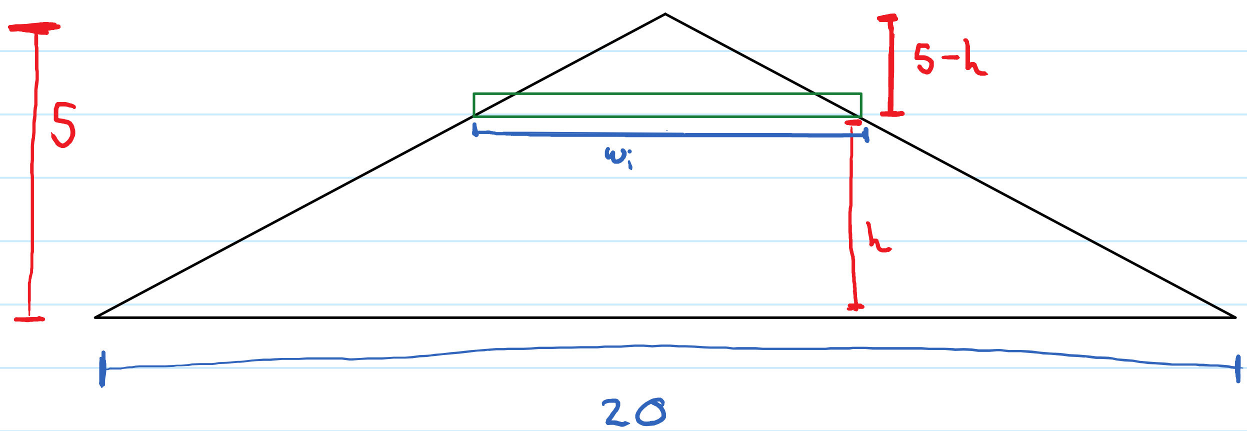
$$= \sum_{i=1}^{5} w_i \Delta h$$

As the number of slices, N, increases, the estimate is more and more accurate. Therefore,

$$(*) \qquad \text{Area} = \lim_{N \to \infty} \left(\sum_{i=1}^{N} w_i \Delta h \right)$$

(This should look familiar!)

Let's describe the width, w_i, of a slice in terms of the height, h.



Note that the small triangle on top is similar to the big triangle. Therefore,

$$\frac{w_i}{20} = \frac{5-h}{5} \Rightarrow \boxed{w_i = 4(5-h)}$$

Plugging this into the eqn (*), we have

$$\text{Area} = \lim_{N \to \infty} \left(\sum_{i=1}^{N} w_i \Delta h \right) = \underbrace{\lim_{N \to \infty} \left(\sum_{i=1}^{N} (20 - 4h) \Delta h \right)}_{\text{def of definite integral!}}$$

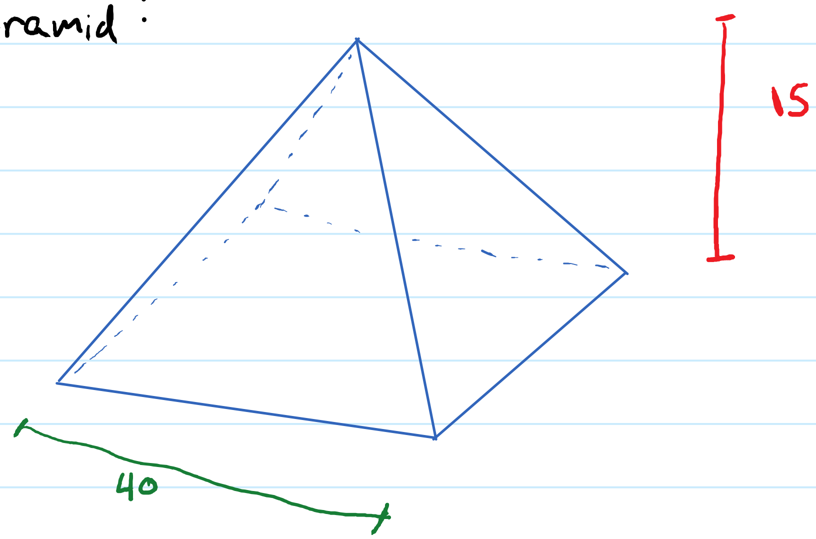
$$= \int_0^5 20 - 4h \, dh.$$

Notice the bounds are from 0 to 5 because h ranges from 0 to 5. Therefore

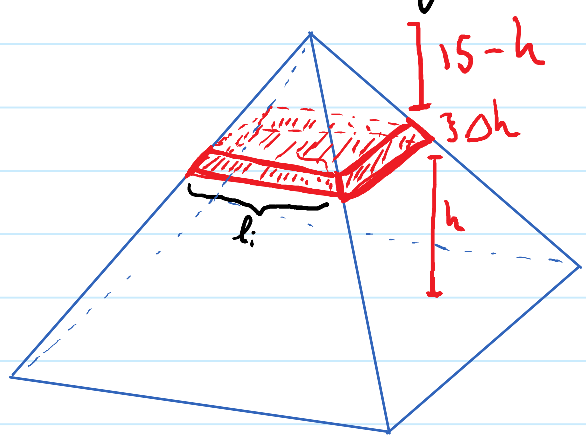
$$\text{Area} = \int_0^5 20-4h\,dh = \left(20h-2h^2\right)\Big|_0^5 = 100-50 = \boxed{50}.$$

From now on, we will skip directly to calculating area or volume of the slices.

<u>Example:</u> ② Find the volume of the square based pyramid:

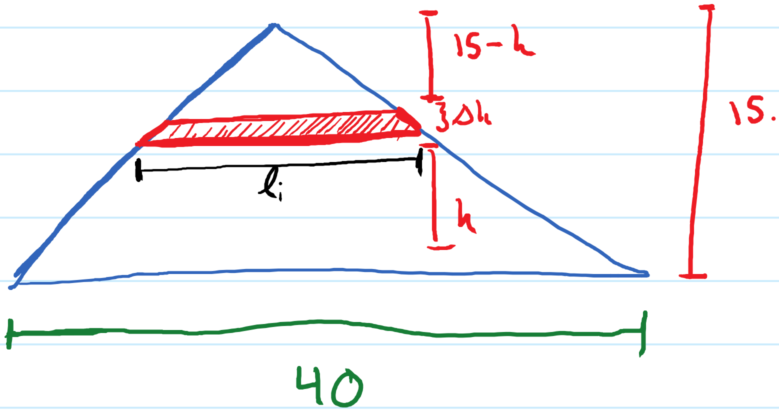


Let's use horizontal slicing:



A_i = red volume
$= l_i^2$

The volume, V_i, of this slice is $\Delta h \cdot A_i$ where A_i is the area of the (square) base.
To find the area of square base, we can look at a vertical cross section of the pyramid.



We are now looking at a side view of the pyramid. l_i is the side length of the square base.

We can again use similar triangles:

$$\frac{l_i}{40} = \frac{15-h}{15} \Rightarrow l_i = \frac{40(15-h)}{15}$$

$$l_i = \frac{8(15-h)}{3}$$

Since $A_i = l_i^2$, $A_i = \left(\frac{8}{3}(15-h)\right)^2$

$$= \frac{64}{9}(225-30h+h^2).$$

Finally, we have:

$$\text{Volume} = \lim_{N\to\infty}\left(\sum_{i=1}^{N} A_i \Delta h\right) \quad (N = \#\text{ of slices})$$

$$= \lim_{N\to\infty}\left(\sum_{i=1}^{N} \frac{64}{9}(225-30h+h^2)\,\Delta h\right) \longleftrightarrow \text{def of integral}$$

$$= \int_0^{15} \frac{64}{9}(225-30h+h^2)\,dh$$

$$= \frac{64}{9}\left(225h - 15h^2 + \frac{1}{3}h^3\right)\Big|_0^{15}$$

$$= \frac{64}{9}\left(\cancel{(225)}(15) - 15\cancel{(15)}^2 + \frac{1}{3}(15)^3\right) - 0$$

$$= \left(\frac{64}{9}\right)\left(\frac{1}{3}\right)(15)^3 = (64)(125) = \boxed{8000}$$