Longo: Math 10B - Winter 2017
Lecture Notes

Date: February 22, 2017

Section

- [ ] §8.1 (cont.)
- [ ] §8.2

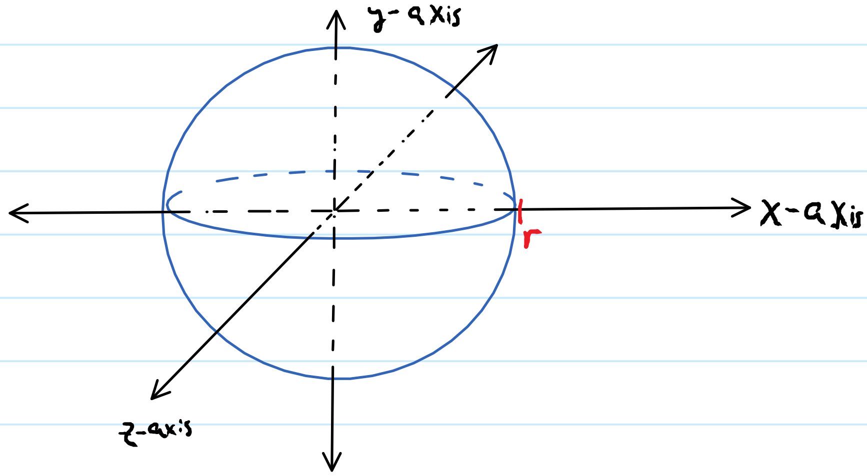
Topics Covered:

- [ ] Volumes of revolution (possibly difficult)
- [ ] Volumes using cross sections (possibly difficult)

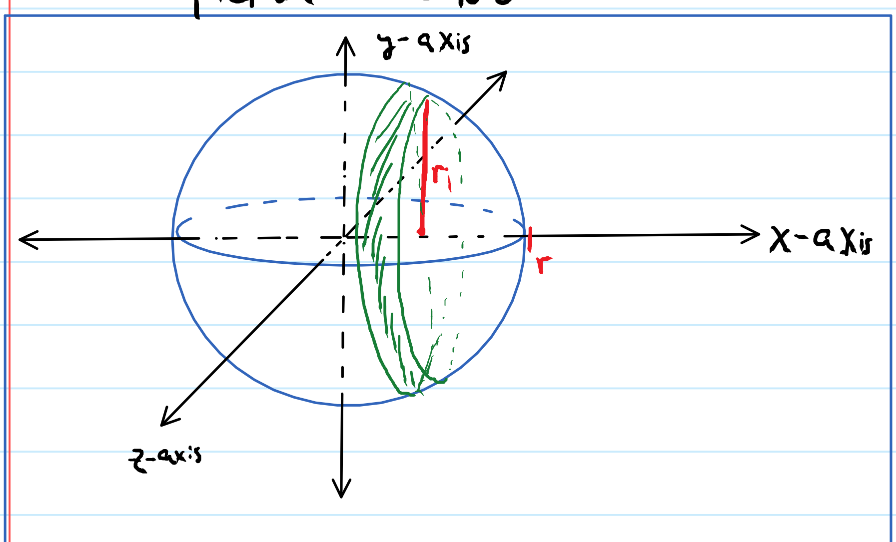
§ 8.1 (cont.):

Example: Use the method of slicing to find a formula for the sphere of radius r.

Sol.



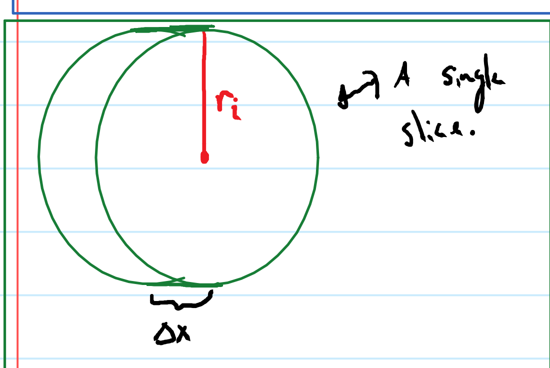
Start by drawing the sphere centered at the origin and imagine a 3rd "z-axis" coming out of the page. Warning: This is not the way we usually orient the axes in 3 dimensions, but we will do it this way for this class. Slice the sphere as in the picture below:



A single slice is a disc with width Δx. The base of the disc is a circle, so the volume, V_i, of the disc is:

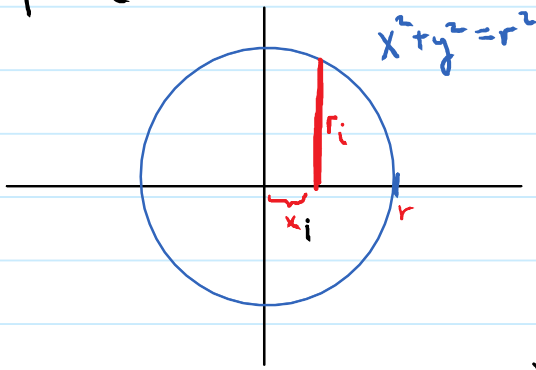
$$V_i = (\text{area of base}) \times (\text{width})$$

$$V_i = \pi r_i^2 \Delta x$$



where r_i is the radius of the circular base.

We need to express r_i as a function of x. To do this, we look at the intersection of the sphere and the xy-plane:



This intersection is the circle w/ radius r, centered at the origin: $x^2+y^2=r^2$.
If x_i is the fixed x-value of the base of the slice, then r_i is just the corresponding (positive) y value. Therefore $r_i^2+x_i^2=r^2 \Rightarrow r_i=\sqrt{r^2-x^2}$.
(Remember: r is fixed). So the area of a single slice is: $V_i=$(area of circular base) $\times$ (width of slice)

$$V_i = \pi r_i^2 \Delta x$$

$$V_i = \pi\left(\sqrt{r^2-x^2}\right)^2 \Delta x$$

$$\boxed{V_i = \pi(r^2-x^2)\Delta x.}$$

When we take the limit as the number of slices, N, goes to infinity, we get:

$$\text{Volume of sphere of radius } r = \lim_{N\to\infty}\left(\sum_{i=1}^{N} V_i\right)$$

$$= \lim_{N\to\infty}\left(\sum_{i=1}^{N}\pi(r^2-x_i^2)\Delta x\right)$$

$$\boxed{= \int_{-r}^{r}\pi(r^2-x^2)\,dx}$$

To get the bounds for x, notice that the x-values on the sphere range from $-r$ to r.

$$\text{Vol} = \int_{-r}^{r}\pi(r^2-x^2)\,dx = \int_{-r}^{r}\pi r^2-\pi x^2\,dx = \left(\pi r^2 x-\tfrac{1}{3}\pi x^3\right)\Big|_{x=-r}^{x=r}$$

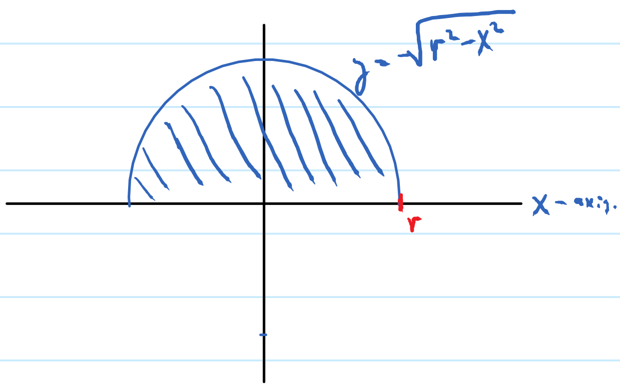
$$= \left(\pi r^2(r) - \tfrac{1}{3}\pi(r^3)\right) - \left(\pi r^2(-r) - \pi(-r)^3\right)$$

$$= \pi r^3 - \tfrac{1}{3}\pi r^3 + \pi r^3 - \tfrac{1}{3}\pi r^3$$

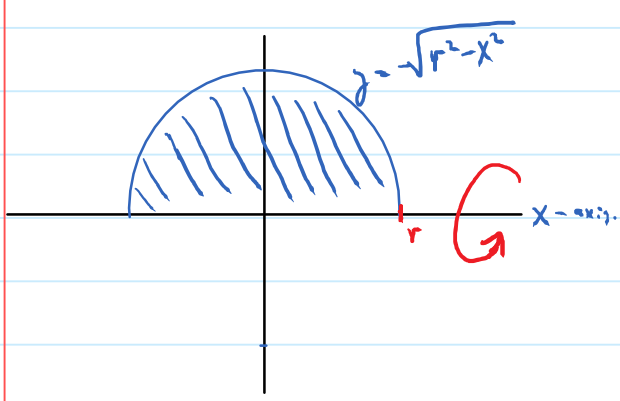
$$= \boxed{\tfrac{4}{3}\pi r^3}$$

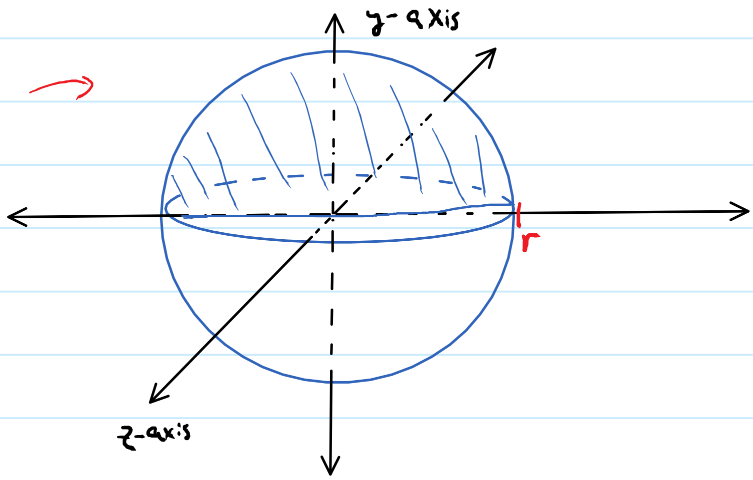
§8.2: Solids of Revolution:

Here is another way to think of the previous example is the following: Start with the graph of the fcn $f(x) = \sqrt{r^2 - x^2}$ and fill in the area under the curve and above the x-axis.

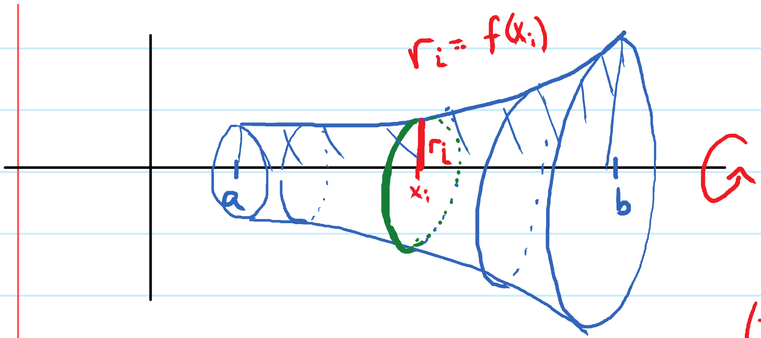


Now imagine you rotate this area about the x-axis (so that the shape comes off the paper towards us.





The shape that gets traced out is the sphere of radius r. If we do this process with any fcn f(x), the resulting solid is called a Solid of revolution.



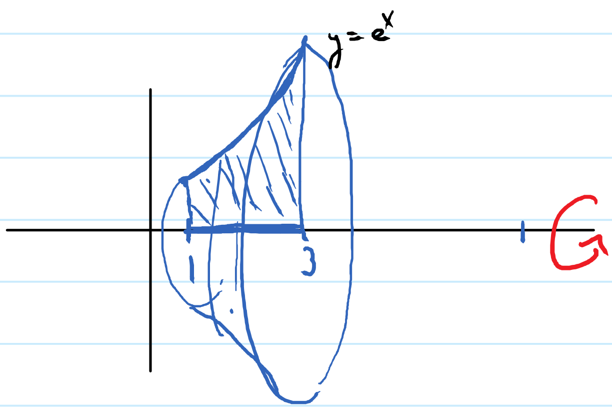
If we use the method of slicing, we see that each slice has Volume $V_i = \pi(f(x_i))^2 \Delta x$ (To see this, use the same exact method we used for the sphere). So we get that

The volume of the solid obtained by revolving the graph of $f(x)$ about the x-axis is:

$$\boxed{\lim_{N\to\infty}\left(\sum_{i=1}^{N} \pi(f(x_i))^2 \Delta x\right) = \pi\int_a^b (f(x))^2\,dx.}$$

Example: ① Find the volume of the solid obtained by revolving the area bound by the curves: $x=1$, $x=3$, $y=0$, and $y=e^x$ about the x-axis.

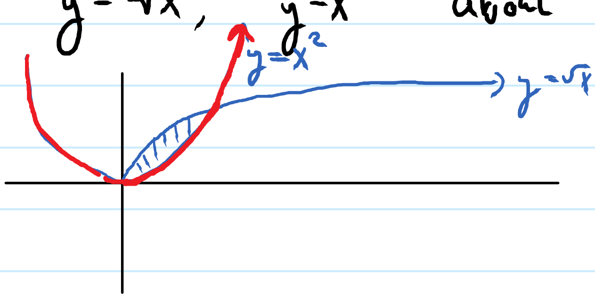
Sol:



Using the formula, we get the volume of the sphere is

$$\pi\int_1^3 (e^x)^2\,dx = \pi\int_1^3 e^{2x}\,dx$$
$$= \frac{\pi}{2}\left(e^{2x}\Big|_1^3\right)$$
$$= \boxed{\frac{\pi}{2}(e^6 - e^2)}$$

Example: ② Find the volume of the solid obtained from rotating the region bounded by the curves $y=\sqrt{x}$, $y=x^2$ about the x-axis.

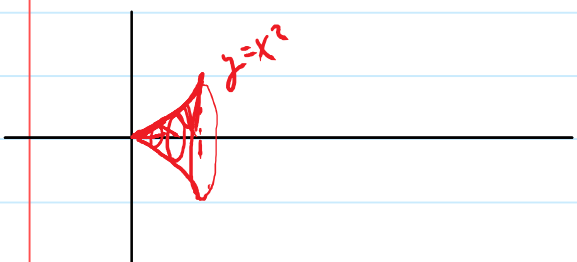


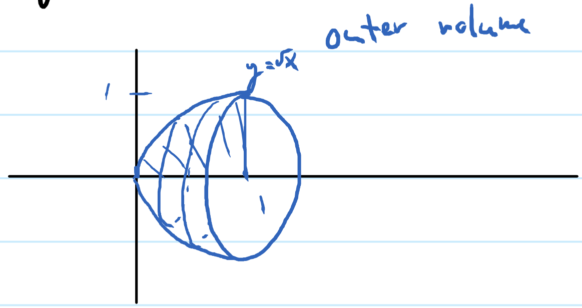
Let's first find the points of intersection:
The two graphs intersect when $y=\sqrt{x}$ and $y=x^2$

$$\Rightarrow \sqrt{x}=x^2$$
$$\Rightarrow x=x^4$$
$$\Rightarrow 0=x^4-x=x(x^3-1)$$
$$\Rightarrow x=0 \text{ or } x=1$$

So the bounds on x from 0 to 1.

To get the volume we want, we will subtract the volume generated by revolving the smaller fcn from the volume we get by revolving the larger fcn.





So the volume is $\underbrace{\pi\int_0^1 (\sqrt{x})^2\,dx}_{\text{outer volume}} - \underbrace{\pi\int_0^1 (x^2)^2\,dx}_{\text{inner volume.}}$

$$= \pi\int_0^1 x - x^4\,dx$$
$$= \pi\left(\frac{x^2}{2}-\frac{x^5}{5}\right)\Big|_0^1$$
$$= \pi\left(\left(\frac{1}{2}-\frac{1}{5}\right)-\left(\frac{0}{2}-\frac{0}{5}\right)\right)$$
$$= \boxed{\pi\left(\frac{3}{10}\right)}$$

<u>Note:</u> We get the formula $\pi\int_a^b (f(x))^2\,dx$ because when we rotate about the x-axis, the radius of the circular slice is $f(x)$. Therefore the area of the circular base was $\pi f(x)^2$. We will see next time that if we revolve the region about a line that is not the x-axis, the formula will

be $\pi\int_a^b (r(x))^2 dx$ where $r(x)$ is still the radius of a circular slice.