Longo: Math 10B - Winter 2017
Lecture Notes

Date: February 24, 2017

Section:

- [ ] §8.2 (cont.)

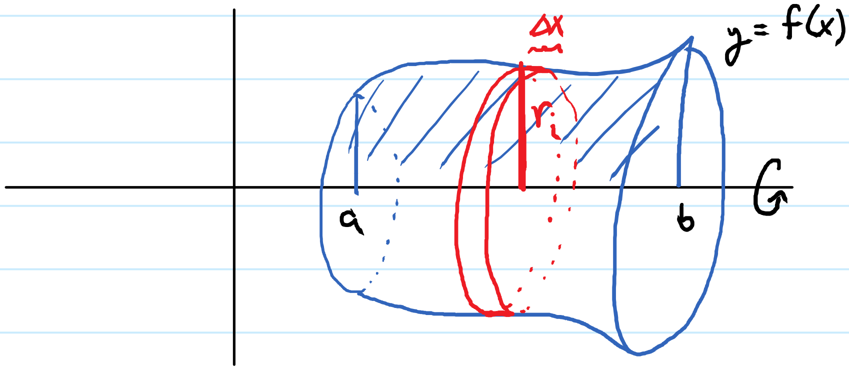
Topics Covered:

- [ ] More solids of revolution

More solids of Revolution:

Last time, we saw that if we start with a region, R, bound by $y = f(x)$, $y = 0$, $x = a$, $x = b$ and revolve it about the x-axis, then the volume of the resulting solid is:

$$\pi \int_a^b f(x)dx.$$



This is because if we slice the solid into discs of width Δx the volume of each disc is

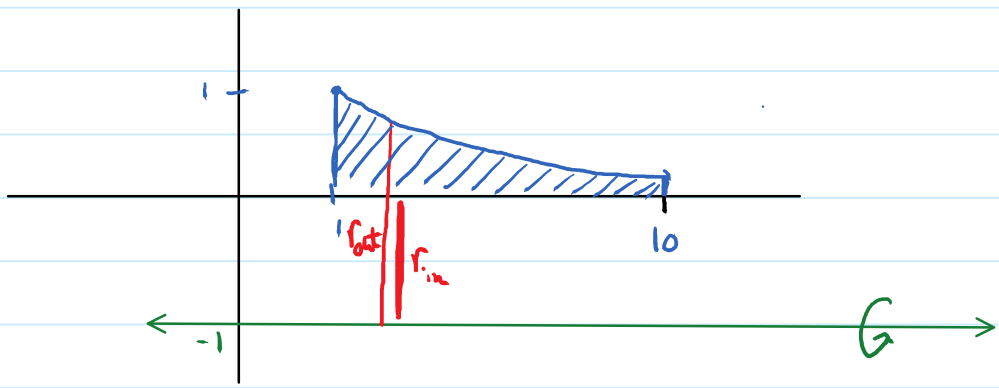
$$V_i = \pi r_i^2 \Delta x$$ where r_i is the radius of the circular base of the disc, and in this specific instance, $r_i = f(x)$. Therefore,

$$Vol = \lim_{N\to\infty} \left(\sum_{i=1}^{N} V_i \right) = \underbrace{\lim_{N\to\infty} \left(\sum_{i=1}^{N} \pi f(x)^2 \Delta x \right)}_{\text{def. of integral}} = \pi \int_a^b f(x)\,dx$$

If we wish to revolve the region, R, about a different axis, the discs would have different radii, so we need to use different formulae.

In every example for the day, we revolve the given region about the given axis.

Examples: (1) R: region bound by $y=\frac{1}{x}$, $x=1$, $x=10$, $y=0$
axis: $y=-1$.



This time, we take the volume of the solid generated by revolving the outside radius, r_{out}, minus the volume of the solid generated by the inner radius, r_{inn}. So our formula will be

$$\int_1^{10} \underbrace{\pi r_{out}^2}_{\text{outer volume}} - \underbrace{\pi r_{in}^2}_{\text{inner volume}}\, dx$$

Here, r_{in} is the distance from the line $y=-1$ to the line $y=0$, so $r_{in}=1$. r_{out} is the distance from $y=\frac{1}{x}$ to the line $y=-1$, so $r_{out}=1+\frac{1}{x}$.

$$\therefore \text{Volume} = \int_1^{10} \pi\left(1+\frac{1}{x}\right)^2 - \pi\, dx$$

$$= \pi \int_1^{10} \left(1+\frac{2}{x}+\frac{1}{x^2}\right) - 1\, dx$$

$$= \pi \int_1^{10} \frac{2}{x}+\frac{1}{x^2}\, dx$$

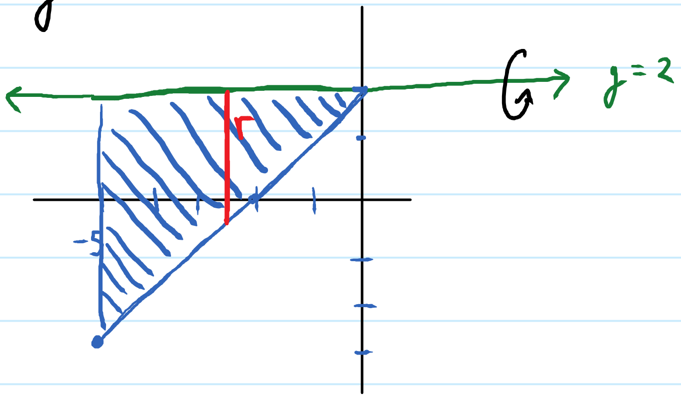
$$= \pi\left(2\ln|x| - \frac{1}{x}\right)\Big|_1^{10}$$

$$= \pi\left(\left(2\ln(10) - \frac{1}{10}\right) - \left(2\ln(1) - 1\right)\right)$$

$$= \boxed{\pi\left(2\ln(10) + 1 - \frac{1}{10}\right)}$$

(2) R: Region bound by $y=0$, $x=-5$, $x=0$
$y=2+x$.

axis: $y=2$:



This time, the radius of a disc is the distance from the line $y=2$ and the line $y=2-x$. Therefore,

$$r = (2-x)-2 = -x.$$

Therefore the volume is:

$$\pi\int_{-5}^{0}(-x)^2\,dx = \pi\int_{-5}^{0}x^2\,dx$$

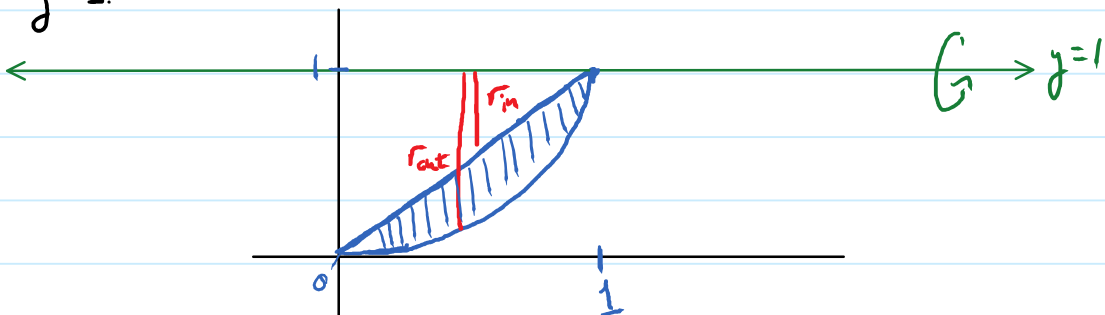
$$= \pi\left(\frac{x^3}{3}\Big|_{-5}^{0}\right)$$

$$= \pi\left(0 - \left(\frac{(-5)^3}{3}\right)\right)$$

$$= \pi\left(-\left(\frac{-125}{3}\right)\right)$$

$$= \boxed{\frac{125\pi}{3}}$$

③ R: region bound by $y=x^2$, $y=x$.
axis: $y=1$.



Like in the first example, the volume will be

$$\pi\int_{0}^{1} r_{out}^2 - r_{in}^2\,dx$$

This time, r_{out} is the distance from the graph $y=x^2$ to the line $y=1$.

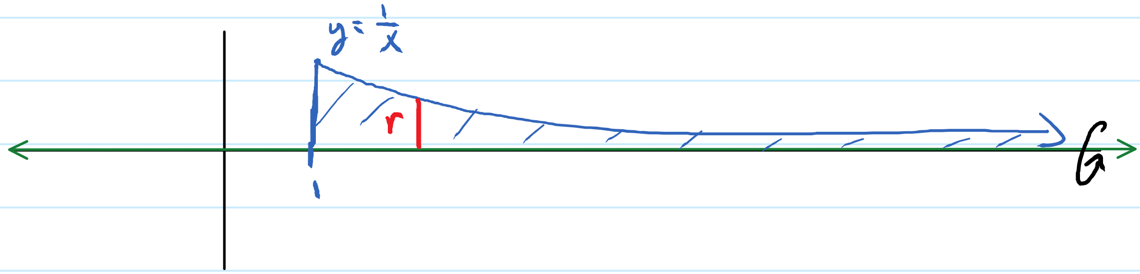
So $r_{out} = 1-x^2$.

Similarly, r_{in} is the distance from the graph of $y=x$ to $y=1$, so $r_{in} = 1-x$.

$$\begin{aligned}
\text{Vol} &= \pi \int_0^1 (1-x^2)^2 - (1-x)^2\,dx \\
&= \pi \int_0^1 \cancel{1}-2x^2+x^4 - (\cancel{1}-2x+x^2)\,dx \\
&= \pi \int_0^1 x^4-3x^2+2x\,dx \\
&= \pi \left(\frac{x^5}{5} - x^3 + x^2\right)\Big|_0^1 \\
&= \pi\left(\left(\frac{1}{5} - 1 + 1\right) - (0)\right) = \boxed{\frac{\pi}{5}}
\end{aligned}$$

4) R: region bound by $x=1$, $y=0$, $y=\frac{1}{x}$

axis: $y=0$ (x-axis).



This time, $r = \frac{1}{x}$. Since the region goes off to ∞, we need an Improper integral.

$\text{Vol} = \pi \int_1^\infty r^2\,dx$. In particular, the solid of revolution could have infinite volume.

$$\text{Vol} = \pi \int_1^\infty \left(\frac{1}{x}\right)^2 dx = \lim_{b\to\infty} \pi \int_1^b \frac{1}{x^2}\,dx$$

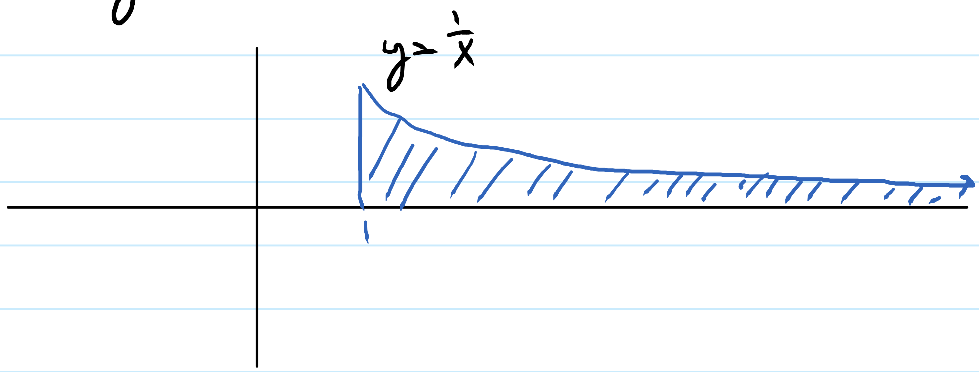
$$= \lim_{b\to\infty} \pi\left(-\frac{1}{x}\Big|_1^b \right)$$

$$= \lim_{b\to\infty} \pi\left(-\frac{1}{b} - (-1) \right)$$

$$= \lim_{b\to\infty} \pi\left(1 - \frac{1}{b} \right) = \boxed{\pi}$$

Q? Did your brain just explode?

It probably should have. We started with a region



That has infinite area (see Lecture 15). However, when we revolve it about the x-axis, we get a solid with finite volume π.

This particular solid is called Gabriel's Horn.