Longo: Math 10B - Winter 2016
Lecture Notes

Date: March 3, 2017

Section:

- [ ] §11.4

Topics Covered:

- [ ] Separable differential equation

§ 11.4: Separation of variables:

Consider the first order differential equation:

$$\frac{dy}{dx} = -\frac{x}{y}.$$

In this equation, y is a function of x, and the derivative of y w.r.t. x at any point is equal to

$$\frac{-x}{y}.$$

$\boxed{Q?}$ How do we find all possible solutions to the diff. eq.?

The above eqn. is a special type of first order diff. eq. called separable. Notice we can "separate" the x and y variables by putting all the ys on one side of the eqn, and all the xs on the other side:

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$\Rightarrow \quad y\frac{dy}{dx} = -x.$$

Then if we "multiply by dx" on both sides, we get

$$y\frac{dy}{dx} = -x$$

$$\Rightarrow \quad y\,dy = -x\,dx$$

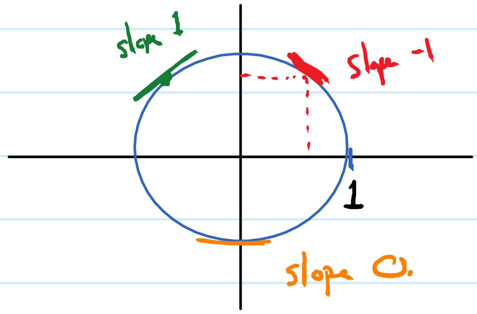
Then by integrating both sides:

$$\int y\,dy = -\int x\,dx$$

$\Rightarrow \quad \frac{y^2}{2} = -\frac{x^2}{2} + C$ for some constant C.

$\Rightarrow x^2+y^2 = 2C$. Set $D = 2C$.

Therefore, any solution satisfies the eqn $y^2+x^2 = D$ for some constant D. These are circles of radius $\sqrt{D}$



Solution with $D=1$.

In this picture, we can check that the slope of the tangent line at any point should be $\frac{-x}{y}$.

Ex: • at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ $\frac{dy}{dx} = \left(\frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}\right) = -1$ ✓

• at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ $\frac{dy}{dx} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$ ✓

• at $(0,-1)$, $\frac{dy}{dx} = \frac{0}{-1} = 0$ ✓

We can also check that $y^2+x^2 = 1$ is a solution by using implicit differentiation.

$$y^2+x^2 = 1$$

$$\Rightarrow \frac{d}{dx}(y^2+x^2) = \frac{d}{dx}(1)$$

$$\Rightarrow 2y\frac{dy}{dx} + 2x = 0$$

$$\Rightarrow 2y\frac{dy}{dx} = -2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$ ✓.

$\boxed{\text{Rmk:}}$ In some sense, we can think of separation of variables as the "anti" implicit differentiation method.

Ex: Find the general solution to the separable first order diff. eq:

$$\frac{dy}{dx} = (0.1)y \qquad \text{(ex. 1 from last time).}$$

Sol: Separate variables, and then integrate both sides.

$$\frac{dy}{dx} = (0.1)y$$
$$\Rightarrow \quad \left(\frac{1}{y}\right)\left(\frac{dy}{dx}\right) = (0.1)$$
$$\Rightarrow \quad \frac{1}{y}\,dy = (0.1)\,dx$$
$$\Rightarrow \quad \int \frac{1}{y}\,dy = \int (0.1)\,dx$$
$$\Rightarrow \quad \ln|y| = (0.1)x + C.$$

In this example, we can solve for y.

$$\Rightarrow \quad |y| = e^{(0.1)x + C} = e^{(0.1)x} e^{C}$$

Since C is an arbitrary constant, $A = e^{C}$ is an arbitrary positive constant.

So $\quad |y| = A e^{(0.1)x}$ for some positive constant A.

$$\Rightarrow \quad y = (\pm A)\, e^{(0.1)x}$$

Again since A is an arbitrary positive constant, $\pm A = B$ can be any real # (warning: 0 does not arise as $\pm e^{C}$ for any real # C. Nevertheless, one can check that $B = 0$ is a sol'n since $y = 0$ satisfies the diff. eq.).

Therefore, any solution must be of the form

$$y = Be^{(0.1)x} \quad \text{for some real \# } B.$$

Ex: (a) Find the general solution to the diff. eq.

$$2\frac{du}{dt} = u^2$$

(b) Solve the initial value problem

$$2\frac{du}{dt} = u^2 \; ; \qquad u(2) = 9$$

Sol: (a) Separate the variables:

$$2\frac{du}{dt} = u^2$$

$$\Rightarrow \quad \frac{2}{u^2}\frac{du}{dt} = 1$$

$$\Rightarrow \quad \frac{2}{u^2}\,du = 1\,dt$$

$$\Rightarrow \quad \int \frac{2}{u^2}\,du = \int 1\,dt$$

$$\Rightarrow \quad \frac{-2}{u} = t + C$$

$$\Rightarrow \quad -2 = u(t+C)$$

$$\Rightarrow \quad \frac{-2}{t+C} = u.$$ for some constant C.

(b) $u(2)=9$ means if you plug in $t=2$, $u=9$.

$$\Rightarrow \quad 9 = \frac{-2}{2+C}$$

$$\Rightarrow \quad 2+C = \frac{-2}{9}$$

$$\Rightarrow \quad C = \frac{-2}{9} - 2$$

$$\boxed{C = \frac{-20}{9}}$$

So the solution is $\boxed{u = \frac{-2}{t - \frac{20}{9}} = \frac{-18}{9t-20}}$

Ex: Solve the initial value problem

$$\frac{dz}{dt} = z + zt^2 \quad \text{such that} \quad z=5 \quad \text{when } t=0.$$

Sol: Separate the variables and integrate!

$$\frac{dz}{dt} = z + zt^2$$

$$\Rightarrow \frac{dz}{dt} = z(1+t^2)$$

$$\Rightarrow \frac{1}{z}\cdot\frac{dz}{dt} = 1+t^2$$

$$\Rightarrow \frac{1}{z}\cdot dz = (1+t^2)dt$$

$$\Rightarrow \int \frac{1}{z}\, dz = \int 1+t^2 dt$$

$$\Rightarrow \ln|z| = t + \frac{1}{3}t^3 + C$$

$$\Rightarrow |z| = e^{t+\frac{1}{3}t^3+C}$$

$$\Rightarrow |z| = e^{t+\frac{1}{3}t^3} e^C$$

$$\Rightarrow z = (\pm e^C)\, e^{t+\frac{1}{3}t^3}$$

Since C is an arbitrary constant, so is $\pm e^C$. If we rename $B = \pm e^C$, the general solution is

$$z = B e^{t+\frac{1}{3}t^3} \quad \text{where } B \text{ is some real \#.}$$

To find B, we use the initial value: $z=5$ when $t=0$.

$$\Rightarrow 5 = Be^0$$

$$\Rightarrow \boxed{5 = B}$$

$\therefore$ The solution to the initial value problem is

$$\boxed{z = 5e^{t+\frac{1}{3}t^3}.}$$