Longo: Math 10B - Winter 2017
Lecture Notes

Date: March 6, 2017

Section:

- §11.5

Topics Covered:

- An application of differential equations: growth and decay

§ 11.5: Word Problems - Exponential growth and decay:

Example: Suppose A population of deer suffers a continuous population decline of 5% of the total population per year. If P is the fcn that tells us the deer population after t-years, we can model this situation using the first order differential eqn:

$$\frac{dP}{dt} = -(0.05)P$$

We've seen this type of diff. eq. before. To solve, we separate the variables and integrate on both sides.

$$\frac{1}{P}\frac{dP}{dt} = -0.05$$
$$\Rightarrow \frac{1}{P}\,dP = -0.05\,dt$$
$$\Rightarrow \int \frac{1}{P}\,dP = \int (-0.05)\,dt$$
$$\Rightarrow \ln|P| = -0.05t.$$

We saw last time that if you solve for P, we get the general solution

$$P = Be^{-0.05t}$$

where B is a real #. In fact, since P represents the population of a colony of deer, we can further assume $P > 0$.

Terminology: Here -0.05 is called the relative growth rate and $\frac{dP}{dt} = -0.05P$ is called the absolute growth rate.

We can generalize our findings with:

Thm: Let k be any (nonzero) real #.
Any solution to the diff. eqn:

$$\frac{dy}{dt} = ky$$

has the form: $y(t) = Be^{kt}$.

This diff. eq. models exponential decay when $k<0$, and it models exponential growth when $k>0$.

Notice that when $t=0$, we get $y(0) = Be^{k\cdot 0} = Be^{0} = B$. If we think of t as time, this tells us B is the initial quantity. I.e., B is the amount at time $t=0$.

Example: Suppose you open a bank account. The bank offers a continuous interest rate of 10%/yr. Suppose you initially deposit \$1,000. Write the initial value diff. eq. that models the growth from interest. If you do not withdraw or deposit any money, how long will it take your initial investment to triple?

Sol: Since the relative continuous growth rate of 10%, the absolute growth rate is

$$\frac{dM}{dt} = (0.1)M$$ where M represents the money in the account.

Since we have the initial deposit amount of \$1,000, we get the initial value diff. eq.:

$$\frac{dM}{dt} = (0.1)M; \quad M(0) = 1{,}000$$

By the above, we can therefore say

$$M(t) = 1{,}000\, e^{(0.1)t}.$$

If we want to know hol long it would take for our initial investment to triple, we solve for t in the equation

$$3{,}000 = 1{,}000\, e^{(0.1)t}$$

$$\Rightarrow \quad 3 = e^{(0.1)t}$$

$$\Rightarrow \quad \frac{\ln(3)}{0.1} = t$$

$$\Rightarrow \quad \boxed{t \approx 10.9} \text{ years}$$

<u>Newton's Law of Cooling</u>: Isaac Newton proposed that a hot or cold object will cool/heat at a rate that is proportional to the difference between the temp of the object and it's surroundings. We can therefore model cooling and heating with a differential eqn.

<u>Example</u>: Suppose a $70^\circ C$ cup of coffee is placed on the counter in a room where the temperature is $15^\circ C$. Suppose after 10 minutes the coffee is $60^\circ C$. Find a diff. eqn that models the change in temp. the coffee. Find a fcn for the temp of the coffee. Graph the fcn.

<u>Sol.</u> Let $C(t)$ be the temp. of the coffee at time t. By Newton's Law of Cooling, $\frac{dC}{dt}$ is proportional to the difference between the coffee's current temp. and its environment. So

$$\frac{dC}{dt} = k(15 - C) \quad \text{for some constant } k.$$

We also have $C(10) = 60°$ and $C(0) = 70°$. If we solve this initial value diff. eq., we get

$$\frac{dC}{dt} = k(C - 15)$$

$$\Rightarrow \frac{1}{C-15}\frac{dC}{dt} = k$$

$$\Rightarrow \frac{1}{C-15}\,dC = k\,dt$$

$$\Rightarrow \ln|C-15| = kt + A \quad \text{where } A \text{ is any real.}$$

$$\Rightarrow |C-15| = e^{A}e^{kt}$$

$$\Rightarrow C-15 = (\pm e^{A})e^{kt}$$

Let $B = \pm e^{A}$. Then we get

$$C = Be^{kt} + 15 \quad \text{where } B \text{ is any real \# (including 0!)}.$$

Now to get B and k, we use the fact that

$$C(0) = 70, \quad C(10) = 60.$$

Using $C(0) = 70$, we have

$$70 = Be^{k \cdot 0} + 15$$

$$\Rightarrow 70 = B + 15$$

$$\Rightarrow \boxed{B = 55}$$

(This is the initial difference in temp!)

Now using $C(10) = 60°$, we get

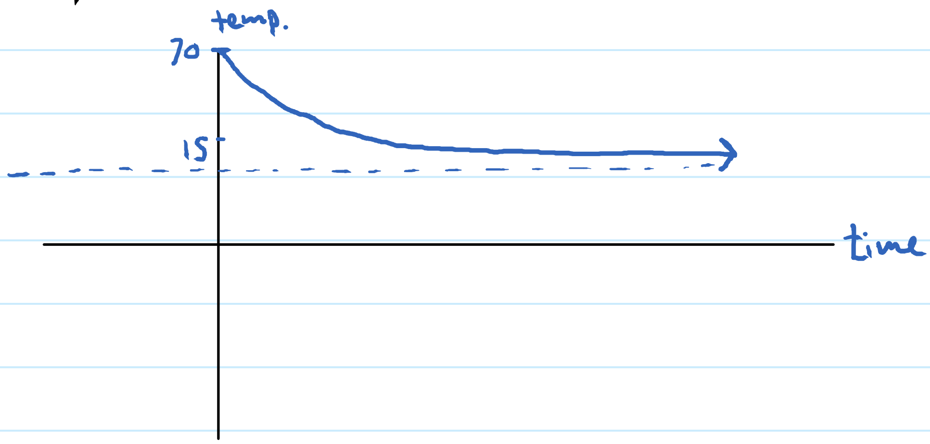
$$60 = C(10) = 55e^{k \cdot 10} + 15$$

$$\Rightarrow \frac{45}{55} = e^{k10}$$

$$\Rightarrow \frac{1}{10}\cdot\ln\left(\frac{9}{11}\right) = k$$

$$\Rightarrow \boxed{k \approx -0.02}$$

Therefore, $C(t) = 55\, e^{-0.02t} + 15.$



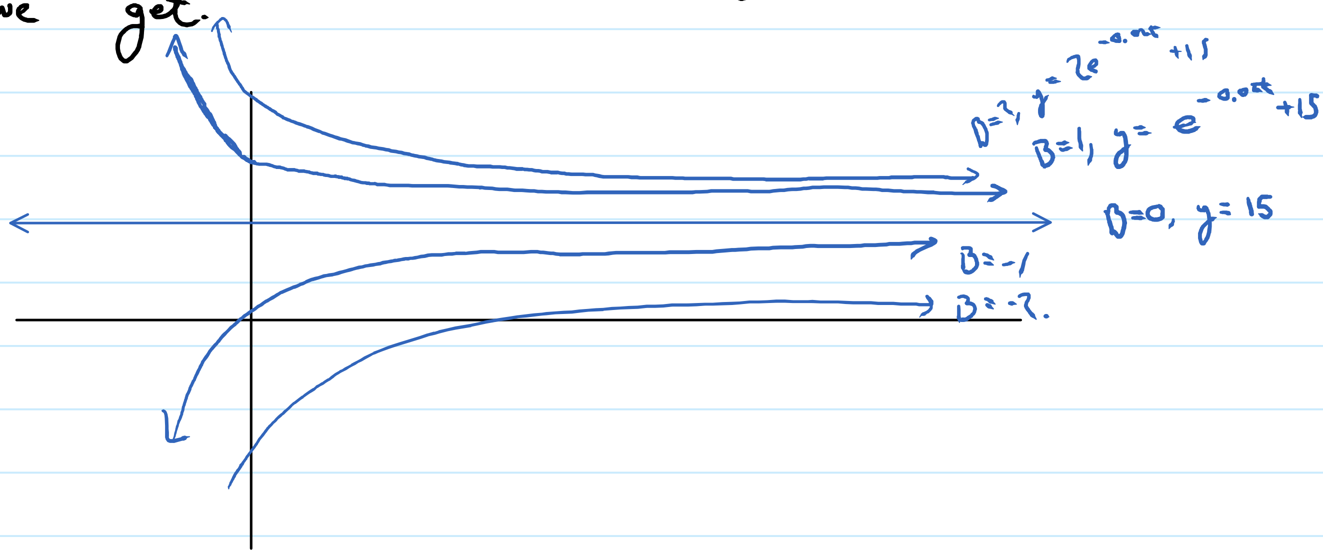
Equilibrium Solutions:

Consider the diff. eq. $\frac{dy}{dt} = -0.02(y-15)$

We saw that the general solution to this diff. eq.

is $y = Be^{-0.02t} + 15$ where B is a real #.

If we let B vary and graph several solutions, we get:



For $B > 0$, the curve lies above the line $y = 15$.
For $B < 0$, the curve lies below the line $y = 15$.

For $B=0$, we get the line $y=15$.

Notice that the solution $y=15$ does not depend on the input variable t. For this reason, $y=15$ is called an equilibrium solution.

Notice also, that as $t \to \infty$, all other solutions approach $y=15$. For this reason $y=15$ is called a stable equilibrium solution.

If, for example, all other solutions depart from $y=15$ as $t \to \infty$, $y=15$ would be called unstable.

Example by picture: