Longo: Math 10B - Winter 2016
Lecture Notes

Date: March 8, 2017

Section:

- [ ] §11.6

Topics Covered:

- [ ] Applications of differential equations and modeling

§11.6 : Modelling: If we have information on the rate of change of a fcn, we can use differential eqns to model real life situations.

Example: The amount of caffeine in your system decays at a continuous rate of 17% per hour. Suppose you drink one cup of coffee / hr. starting at 6 AM and each cup has 130 mg of caffeine.

① Write an initial value problem modelling this situation.

② How much caffeine will be in your system at 12:00 PM.

Sol: ① Let $C(t)$ be the amount of caffeine in your body t hours after 6AM.

At 6:00 AM, you haven't started drinking coffee yet, so

$$C(0) = 0.$$

Each hour, you intake 130 mg caffiene, while at the same time, 17% of the current caffeine level is being metabolized and excreted.

So we have

change in caffeine = (caffeine in) − (caffeine out)

$$\Rightarrow \boxed{\frac{dC}{dt} = 130 - 0.17C.} \quad ; \quad C(0) = 0.$$

(2) We can solve this (separable) diff. eq. to find the fcn $C(t)$:

$$\frac{dC}{dt} = 130 - 0.17C$$

$$\Rightarrow \quad \frac{1}{130-0.17C}\, dC = dt$$

$$\Rightarrow \quad \int \frac{1}{130-0.17C}\, dC = \int dt$$

$\frac{1}{-0.17} \ln|130-0.17C| = t + A$ for some constant A.

$\Rightarrow \ln|130-0.17C| = -0.17t + B$ where $B = -0.17A$ is some constant

Note, last time we solved for C first, and then we found the constant A. It is also possible to find A now:

We know that when $t=0$, $C=0$ so plugging into the equation:

$$\ln|130-0.17C| = -0.17t + B$$

$$\Rightarrow \quad \ln|130-0.17(0)| = 0 + B$$

$$\Rightarrow \quad \boxed{\ln(130) = B}$$

Now for this particular initial value,

$$\ln|130-0.17C| = -0.17t + \ln(130$$

$$\Rightarrow \quad |130-0.17C| = e^{-0.17t + \ln(130)}$$

$$|130-0.17C| = e^{-0.17t} e^{\ln(130}$$

$$|130-0.17C| = 130\, e^{-0.17t}$$

$$\Rightarrow \quad 130 - 0.17C = 130e^{-0.17t}$$

Note: normally we should say $130 - 0.17C = \pm 130e^{-0.17t}$, but since $C=0$ when $t=0$, we know $130 - 0.17C \neq -130e^{-0.17t}$.

$$\Rightarrow \quad \boxed{C = \frac{1}{0.17}\left(130 - 130e^{-0.17t}\right).}$$

Finally, at 12:00PM ($t=6$ hours after 6AM)

$$C(6) = \frac{1}{0.17}\left(130 - 130e^{(-0.17)(6)}\right)$$
$$\approx 489 \text{ mg of caffeine at } 6\text{AM}$$

Challenge: If you continue this pattern for eternity, how much caffeine will be in your system as $t \to \infty$. I.e., what is the equilibrium solution?

Sol: $$\lim_{t\to\infty} C(t) = \lim_{t\to\infty} \left(\frac{300}{0.17}\right)\left(1 - e^{-0.17t}\right)$$

$$= \boxed{\frac{300}{0.17}.}$$

In the long run, the caffeine level stabilizes at $\frac{300}{0.17}$. This is why $y = \frac{300}{0.17}$ is called the equilibrium solution.

Example: Suppose your bank offers a continuous interest rate of 10% / year. Suppose you withdraw from the account at a continuous rate of \$ 2k / yr.

(A) Find a diff. eq. describing the account, and find the general solution.

(B) Solve the IVP that corresponds to an initial deposit of \$ 30 k

Sol: (A) Let $B(t)$ be the balance of the account after t years (in thousands), then

$$\frac{dB}{dt} = (\text{rate increase}) - (\text{rate decrease})$$

$$\Rightarrow \quad \frac{dB}{dt} = (0.1)B - 2$$

Let's factor out the (0.1).

$$\frac{dB}{dt} = (0.1)\left(B - \frac{10}{0.1}\right)$$

$$\Rightarrow \quad \frac{dB}{dt} = (0.1)(B - 20)$$

We solve this separable eqn as before,

$$\frac{1}{B-20}\frac{dB}{dt} = 0.1$$

$$\frac{1}{B-20}\,dB = 0.1\,dt$$

$$\Rightarrow \quad \int \frac{1}{B-20}\,dB = \int 0.1\,dt$$

$$\Rightarrow \quad \ln|B-20| = 0.1t + C \quad , \quad C \text{ constant.}$$

$$\Rightarrow \quad |B-20| = e^{0.1t+C}$$

$$\Rightarrow \quad |B-20| = e^{C}e^{(0.1t)}$$

$$\Rightarrow \quad B-20 = (\pm e^{C})e^{0.1t}$$

$$\Rightarrow \quad B = (\pm e^{C})e^{0.1t} + 20$$

Rename $A = \pm e^{C}$, then we get

$$B(t) = Ae^{0.1t} + 100$$

(B) If we have an initial deposit of \$30k, then

$B(0) = 30$, So

$$30 = Ae^{(0.1)(0)} + 20$$

$$\Rightarrow \quad 10 = Ae^{0}$$

$$\Rightarrow \quad 10 = A$$

So $\quad B(t) = 10e^{0.1t} + 20$