Longo: Math 10B - Winter 2017
Lecture Notes

Date: March 10, 2017

Section:

- [ ] §9.2

Topics Covered:

- [ ] Geometric series

§9.2: Geometric series!

Whenever we have a sequence of numbers:

$a_1, a_2, a_3, \ldots$ where each a_i is a number,

we can form a series by adding them all up:

$$a_1 + a_2 + a_3 + \ldots$$

Each a_i is called the i^{th} term. Even though we may be adding infinitely many terms, the resulting sum could be finite.

Example: ① $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

in Sigma notation $\sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i$.

We saw earlier in the class that this series is equal to 1.

② $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$

it turns out this series is infinite (i.e., it diverges!)

Why? $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \ldots$

$= 1 + \frac{1}{2} + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \ldots$

You can always collect enough terms to make $\frac{1}{2}$, so the series is $> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$, which is clearly infinite.

③ $3 + 6 + 12 + 24 + 48 + \ldots = (3)(2^0) + (3)(2^1) + (3)(2^2) + \ldots$

In sigma notation: $\sum_{i=0}^{\infty} 3(2)^i$.

This series is clearly infinite.

Examples ①, ③ are a special type of series called geometric series. ② is called harmonic. In this class we will focus on geometric series and Taylor Series (next week).

Definition: Let a, r be real numbers. A finite geometric sequence is a sequence of the form

$$\sum_{i=0}^{n} a \cdot r^i = a + ar + ar^2 + ar^3 + \dots + ar^n.$$

An infinite geometric sequence is one of the form

$$\sum_{i=0}^{\infty} a r^i = a + ar + ar^2 + ar^3 + \dots$$

Warning: It is possible that an infinite geometric sequence is ∞. We will soon determine exactly when a geometric sequence converges.

Example: Compute the finite geometric sequence

$$\begin{aligned}\sum_{i=0}^{4} 96\left(\tfrac{1}{2}\right)^i &= 96 + 96\left(\tfrac{1}{2}\right)^1 + 96\left(\tfrac{1}{2}\right)^2 + 96\left(\tfrac{1}{2}\right)^3 + 96\left(\tfrac{1}{2}\right)^4 \\ &= 96 + 96\left(\tfrac{1}{2}\right) + 96\left(\tfrac{1}{4}\right) + 96\left(\tfrac{1}{8}\right) + 96\left(\tfrac{1}{16}\right) \\ &= 96 + 48 + 24 + 12 + 6 \\ &= \boxed{186}\end{aligned}$$

Problem: We obviously don't want to have to do this calculation over and over again. Is there a better way?

Formula for finite geometric series:

Suppose a, and r are given. For any positive whole number n, let

$$S_n = a + ar + ar^2 + \dots + ar^{n-1}$$ be the sum of the first n terms. We will use a telescoping sum trick. Notice that:

$$r \cdot S_n = r(a + ar + ar^2 + \dots + ar^{n-1}$$
$$= ar + ar^2 + ar^3 + \dots + ar^n .$$

$$\Rightarrow \quad S_n - rS_n = (a + ar + ar^2 + \dots + ar^{n-1}) - (ar + ar^2 + \dots + ar^{n-1} + ar^n)$$
$$= a + ar + ar^2 + \dots + ar^{n-1}) - ar - ar^2 - \dots - ar^{n-1} - ar^n$$
$$= a + (ar - ar) + (ar^2 - ar^2) + \dots + (ar^{n-1} - ar^{n-1}) - ar^n$$
$$= a - ar^n$$
$$= a(1 - r^n) .$$

Now if we solve for S_n:

$$S_n - rS_n = a(1 - r^n)$$
$$\Rightarrow \quad S_n(1 - r) = a(1 - r^n)$$
$$\Rightarrow \quad \boxed{S_n = \frac{a(1 - r^n)}{1 - r}}$$

Examples: (1) From before:

$$S_5 = \sum_{i=0}^{4} 96 \cdot \left(\tfrac{1}{2}\right)^i = 96 + 48 + 24 + 12 + 6.$$

Here, $n = 5$, $a = 96$, $r = \frac{1}{2}$. So by the formula,

$$S_5 = \frac{96(1 - (\frac{1}{2})^5)}{1 - \frac{1}{2}} = \frac{96(1 - \frac{1}{32})}{\frac{1}{2}} = 96\left(\frac{\frac{31}{32}}{\frac{1}{2}}\right)$$
$$= 96\left(\frac{31}{16}\right) = \boxed{186}$$

(2) $2+6+18+54+164+486.$

Let's first find r and a. Note that in a geometric series, the i^{th} term is ar^i and the $i-1^{th}$ term is ar^{i-1}. If we divide any term by the one before it, we get

$$\frac{\cancel{a}r^i}{\cancel{a}r^{i-1}} = \frac{r^i}{r^{i-1}} = \boxed{r}$$

For this reason, r is called the common ratio.

In the above series, we have

$$\frac{486}{164}, \quad \frac{164}{54}, \quad \frac{54}{18}, \quad \frac{18}{6}, \quad \frac{6}{2} \quad \text{all equal } \boxed{3}.$$

$\therefore$ we have $r=3$, and $a=2$ (first term).

and $n=6$ (number of terms)

$$\Rightarrow \boxed{S_6 = \frac{2(1-3^6)}{1-3} = \frac{2(1-3^6)}{-2} = -(1-3^6) = 728}$$

Evaluating infinite geometric series:

Let a, r be given. Then the infinite series

$a+ar+ar^2+ar^3+\ldots$ is equal to $\lim_{n\to\infty}(S_n)$ because we are just adding infinitely many terms.

On the other hand, we know $S_n = \frac{a(1-r^n)}{1-r}$.

$$\Rightarrow \boxed{a+ar+ar^2+\ldots = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}}$$

This limit completely depend on what happens to r^n as $n \to \infty$.

Certainly if $|r|>1$, then $|r^n| \to \infty$ as $n \to \infty$.

if $|r|<1$, then $|r^n| \to 0$ (e.g., $\lim_{n\to\infty} (\frac{1}{2})^n = 0$).

Therefore:

Thm: ① If $|r| \geq 1$ $a + ar + ar^2 + \ldots = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$ is not finite since the limit DNE. In this case, we say the geometric series diverges.

② If $|r|<1$,

$$\boxed{a + ar + ar^2 + \ldots = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}}$$

What happens if $|r|=1$? Then $r = \pm 1$ and the series turns into

$$a + a(1) + a(1)^2 + \ldots = a + a + a + \ldots = \pm\infty \text{ (diverges)}$$

or

$$a + a(-1) + a(-1)^2 + a(-1)^3 + \ldots$$
$$= a - a + a - a + a - a \pm \ldots$$

In this case, the series oscillates between a and 0 as you add more terms. Since it doesn't "settle down", we say it diverges.

Ex: ① $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$.

Here, $a = \frac{1}{2}$ (first term)

$r = \frac{1}{2}$ (common ratio)

Since $|r|<1$, the series is finite and equals

$$\frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = \boxed{1}$$

(See notes from 2-13!)

② $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots$

Here, $a = \frac{1}{2}$

$r = -\frac{1}{2}$, so $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots = \frac{\frac{1}{2}}{1-(-\frac{1}{2})} = \frac{\frac{1}{2}}{\frac{3}{2}} = (\frac{1}{2})(\frac{2}{3}) = \boxed{\frac{1}{3}}$