Longo: Math 10B - Winter 2016
Lecture Notes

Date: March 13, 2017

Section:
- [ ] §9.2
- [ ] §10.2

Topics Covered:
- [ ] Geometric series (cont.)
- [ ] Taylor Polynomials

Geometric Series (Cont'd)

From last time Let a, r be whole #s and let
n be a whole number. Last time we saw

$$\boxed{a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r}}.$$

If $|r|<1$, we can also compute the infinite geometric series:

$$\boxed{a + ar + ar^2 + ar^3 + \ldots = \frac{a}{1-r}}$$

Warning: If $|r| \geq 1$, then the infinite series diverges: it either goes off to ∞, or it perpetually oscillates.

Ex: ① Decide if $20 - 10 + 5 - \frac{5}{2} + \frac{5}{4} \mp \ldots$ Converges. If so, calculate it's value.

Sol: Let's find the common ratio, r, by taking any term and dividing by the previous term.
For example:

$$\boxed{r = \frac{5}{-10} = -\frac{1}{2}.}$$

Since $|r| = |-\frac{1}{2}| = \frac{1}{2} < 1$, the infinite geometric series Converges. Since $a = (\text{the first term})$
$= 20$

we have
$$20 - 10 + \frac{5}{2} - \frac{5}{4} \pm \ldots = \frac{a}{1-r}$$
$$= \frac{20}{1-(-\frac{1}{2})}$$
$$= \frac{20}{1+\frac{1}{2}}$$
$$= \frac{20}{\frac{3}{2}}$$
$$= \boxed{\frac{40}{3}}$$

Example: ② Show that the infinitely repeating decimal $0.9999\ldots$ is equal to 1.

Sol: Let's rewrite the decimal as:

$$0.9999\ldots = 0.9 + 0.09 + 0.009 + 0.0009 + \ldots$$
$$= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \ldots$$
$$= 9\left(\frac{1}{10}\right) + 9\left(\frac{1}{10}\right)^2 + 9\left(\frac{1}{10}\right)^3 + \ldots$$

The last expression is an infinite geometric sequence where $a = (\text{first term}) = \frac{9}{10}$ and

$$r = \text{Common ratio} = \frac{\cancel{9}\left(\frac{1}{10}\right)^2}{\cancel{9}\left(\frac{1}{10}\right)} = \frac{1}{10}.$$

Using the formula:

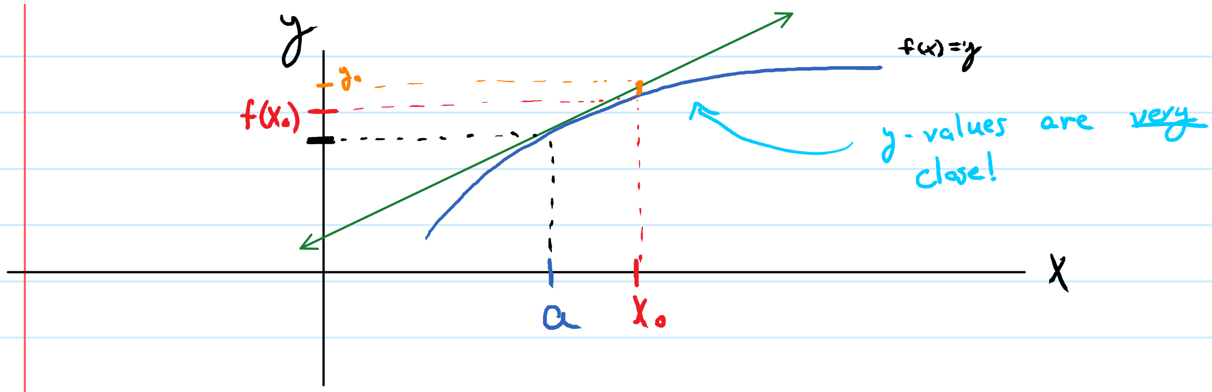
$$9\left(\frac{1}{10}\right) + 9\left(\frac{1}{10}\right)^2 + \ldots = \frac{a}{1-r}$$
$$= \frac{\frac{9}{10}}{1-\frac{1}{10}}$$
$$= \frac{\left(\frac{9}{10}\right)}{\left(\frac{9}{10}\right)} = \boxed{1}$$

✓

§10.1: Taylor Polynomials:

Review from Calc. I: Let $f(x)$ be a diff'ble fcn, and let a be a real # in the domain of $f(x)$.

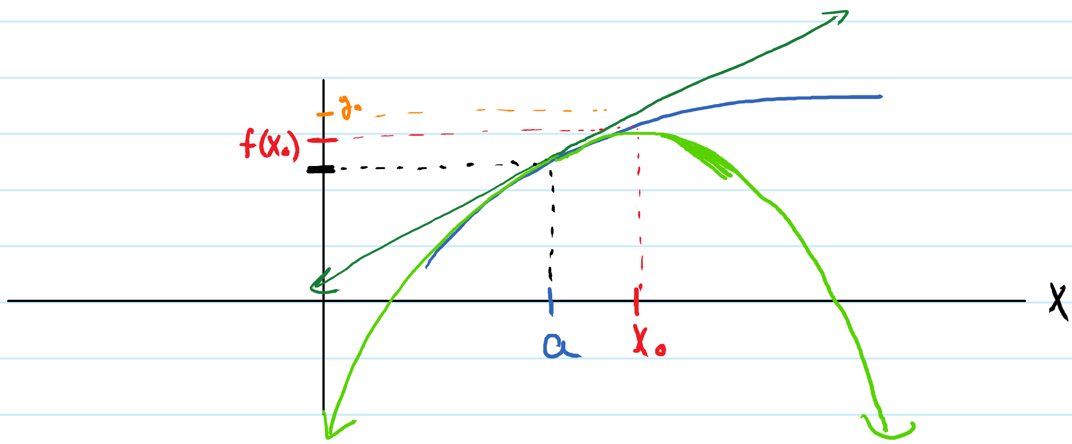
Idea: If x_0 is a # near a, then instead of the value of $f(x_0)$, we can approximate $f(x_0)$ by finding the y-value of the point on the tangent line at f at a with x-value x_0. This is called linear approximation.



Since the equation of the tangent line at a is
$$y = f(a) + f'(a)(x-a) \quad \text{(by "point-slope" formula)}$$

we have $\boxed{f(x_0) \approx f(a) + f'(a)(x-a)}$

Idea: We can get an even better approximation for values of f near a by approximating with a parabola instead of a line.



The parabola stays closer to the graph for longer. The y-values on the parabola approximate the fcn better.

Formula for the parabola: To find a formula for the fcn that defines the parabola, keep in mind we want a degree 2 polynomial fcn such that:

$P_2(x)$

① $P_2(a) = f(a)$
② $P_2'(a) = f'(a)$
③ $P_2''(a) = f''(a)$

You can check that $\boxed{P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.}$

$P_2(x)$ is called the Second order Taylor Polynomial of f, centered at $x=a$.

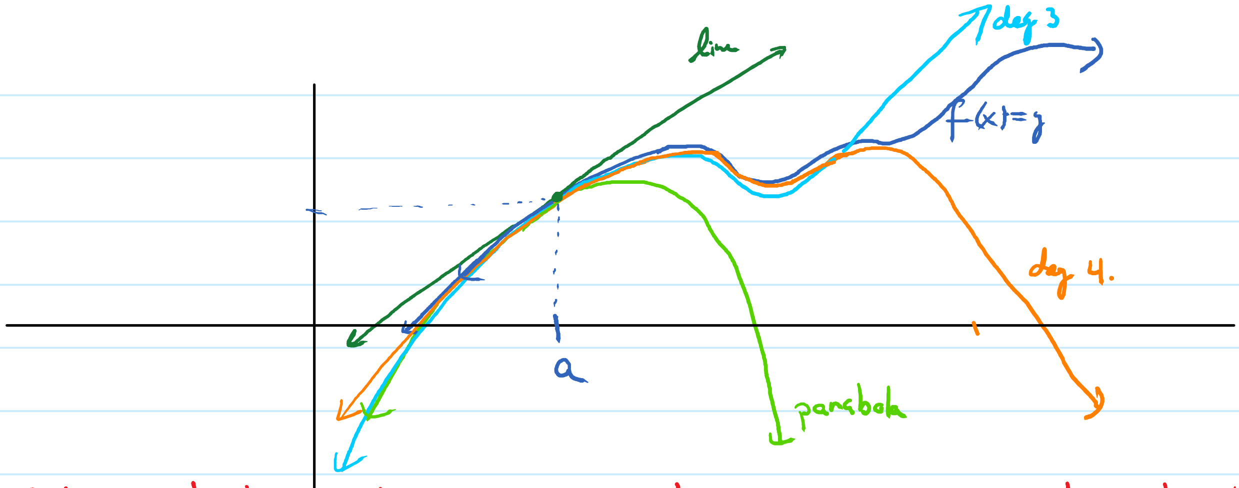
Ex: Find the 2^{nd} order Taylor Poly. of f centered at $x=1$, where $f(x) = \ln(x)$. and approximate $\ln(1.1)$.

Sol:
$$\left.\begin{aligned} f(x) &= \ln(x) \\ f'(x) &= \frac{1}{x} \\ f''(x) &= \frac{-1}{x^2} \end{aligned}\right\} \Longrightarrow \begin{aligned} f(1) &= \ln(1) = 0 \\ f'(1) &= 1 \\ f''(1) &= -1 \end{aligned}$$

So $P_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2$
$$\boxed{P_2(x) = (x-1) - \tfrac{1}{2}(x-1)^2.}$$

Lastly,
$$\begin{aligned} \ln(1.1) = f(1.1) &\approx (1.1-1) - \tfrac{1}{2}(1.1-1)^2 \\ &= (0.1) - \tfrac{1}{2}(0.1)^2 \\ &= 0.1 - \tfrac{1}{2}(0.01) \\ &= 0.1 - 0.005 \\ &= \boxed{0.995} \end{aligned}$$

$\boxed{Q?}$ Why stop at parabolas? If we try to approximate $f(x)$ with a degree 3 polynomial, the approximation will be better. We could go further and even use degree 4 or higher polynomials.



The higher degree of the polynomial, the better the approximation is! This is extremely useful because polynomials are simple and easy to work with. In many cases, we can just work with the Taylor Polys. instead of the original fcn, which makes our lives much easier.

Notation: If $f(x)$ is a fcn, and n is a whole number, then $f^{(n)}(x)$ is the n^{th} derivative of f. I.e., $f^{(n)}(x)$ is the fcn you get when you derive $\underbrace{f, \text{ then derive } f', \text{ then derive } f''(x), \ldots}_{n\text{-times}}$

Def: The n^{th} order Taylor Polynomial of f centered at $x=a$ is the degree n polynomial fcn

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{k=0}^{n} \left(\frac{f^{(k)}(a)}{k!}\right)(x-a)^k.$$

where $k! = (k)(k-1)(k-2)\cdots(3)(2)(1)$ (k factorial).

Remarks: ① If $x \approx a$, $P_n(x) \approx f(x)$

② $f(a) = P_n(a)$
$f'(a) = P_n'(a)$
$f''(a) = P_n''(a)$
$\vdots$
$f^{(n)}(a) = P_n^{(n)}(a).$

③ The Taylor Poly. Centered at $x=0$ is sometimes called a McLaurin Polynomial.

More examples next time: