Longo: Math 10B - Winter 2017
Lecture Notes

Date: March 15, 2017

Section:

- §10.1 (cont.)

Topics Covered:

- Some important examples of Taylor Polynomials

§10.1: Taylor Polynomials:

From last time: Let f be a "nice" fcn, and let a be a point in the domain of f.

Notation: If n is a whole number:
① $f^{(n)}(x)$ is the n^{th} derivative of f.
② $n! = (n)(n-1)(n-2)\cdots(3)(2)(1)$ (n factorial).

The Taylor Polynomial of degree n centered at $x=a$ is the polynomial fcn:

$$\boxed{P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n.}$$

Rmk: ① We use $P_n(x)$ to approximate $f(x)$ for x near a.
② If $a=0$, $P_n(x)$ is also called the degree n McLaurin polynomial.

Some common Taylor Polynomials:

① Calculate the degree 4 Taylor poly for $f(x)=e^x$ near $x=0$.

Sol: Use the formula:

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

(Since $a=0$, $x-a = x-0 = x$. So instead of $(x-a)^k$, we just write x^k)

Since $f(x)=e^x$, $f'(x)=e^x$, $f''(x)=e^x$, $f^{(3)}(x)=e^x$, $f^{(4)}(x)=e^x$
$f(0)=e^0=1$, $f'(0)=1$, $f''(0)=1$, $f^{(3)}(0)=1$, $f^{(4)}(0)=1$.

$$\Rightarrow P_4(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4$$

$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3\cdot 2} + \frac{x^4}{4\cdot 3\cdot 2\cdot 1}$$

$$\boxed{P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}}$$

② Find the degree 5 Taylor Poly. of $g(x) = \sin(x)$ near 0.

Sol: We need the first 5 derivatives of $g(x)$.

This pattern repeats!

$$\begin{aligned} g(x) &= \sin(x) \\ g'(x) &= \cos(x) \\ g''(x) &= -\sin(x) \\ g^{(3)}(x) &= -\cos(x) \\ g^{(4)}(x) &= \sin(x) \\ g^{(5)}(x) &= \cos(x) \end{aligned} \Rightarrow \begin{aligned} g(0) &= \sin(0) = 0 \\ g'(0) &= \cos(0) = 1 \\ g''(0) &= -\sin(0) = 0 \\ g^{(3)}(0) &= -\cos(0) = -1 \\ g^{(4)}(0) &= \sin(0) = 0 \\ g^{(5)}(0) &= \cos(0) = 1 \end{aligned}$$

Note: Every even derivative is 0, the odd derivatives alternate between 1, -1.

$$P_5(x) = g(0) + g'(0)x + \frac{g''(0)}{2}x^2 + \frac{g^{(3)}(0)}{3\cdot 2}x^3 + \frac{g^{(4)}(0)}{4\cdot 3\cdot 2\cdot 1}x^4 + \frac{g^{(5)}(0)}{5\cdot 4\cdot 3\cdot 2\cdot 1}x^5$$

$$P_5(x) = 0 + 1x + \frac{0}{2}x^2 + \frac{(-1)}{6}x^3 + \frac{0}{24}x^4 + \frac{1}{120}x^5$$

$$\boxed{P_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}}$$

③ Find the degree 5 Taylor Poly. of $h(x) = \cos(x)$ near 0.

We need the first 5 derivatives of h

This pattern repeats!

$$\begin{aligned} h(x) &= \cos(x) \\ h'(x) &= -\sin(x) \\ h''(x) &= -\cos(x) \\ h^{(3)}(x) &= \sin(x) \\ h^{(4)}(x) &= \cos(x) \\ h^{(5)}(x) &= -\sin(x) \end{aligned} \Rightarrow \begin{aligned} h(0) &= \cos(0) = 1 \\ h'(0) &= -\sin(0) = 0 \\ h''(0) &= -\cos(0) = -1 \\ h'''(0) &= \sin(0) = 0 \\ h^{(4)}(0) &= \cos(0) = 1 \\ h^{(5)}(0) &= -\sin(0) = 0 \end{aligned}$$

Note: Every odd derivative is 0, and even derivatives bounce between 1 and -1.

$$P_5(x) = h(0) + h'(0)x + \frac{h''(0)}{2}x^2 + \frac{h^{(3)}(0)}{3\cdot 2\cdot 1}x^3 + \frac{h^{(4)}(0)}{4\cdot 3\cdot 2\cdot 1}x^4 + \frac{h^{(5)}(0)}{5\cdot 4\cdot 3\cdot 2\cdot 1}x^5$$

$$P_5(x) = 1 + 0\cdot x + \frac{(-1)}{2}x^2 + \frac{0}{6}x^3 + \frac{1}{24}x^4 + 0x^5$$

$$\boxed{P_5(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}}$$

Rmk Since the derivatives of e^x, $\cos(x)$, $\sin(x)$ repeat, it is easy to generalize what we just did. For example, if we want the order 10 Taylor polys. near 0, we have:

$$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \dots + \frac{x^{10}}{10!}$$

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

$$\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}$$

Side note: Let i be the imaginary unit. I.e., i is the complex number such that $\boxed{i^2 = -1.}$ An extremely famous equation of Euler asserts

$$\boxed{e^{ix} = \cos(x) + i\sin x}$$

Why? Use Taylor Polys! replace both sides of the equation by the degree 5 Taylor Poly:

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!}$$

$$= 1 + ix + \frac{i^2x^2}{2} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!}$$

$$= 1 + ix - \frac{x^2}{2} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!}$$

Meanwhile, $\cos(x) + i\sin(x) =$

$$\left(1 - \frac{x^2}{2} + \frac{x^4}{4!}\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)$$

$$= 1 + ix - \frac{x^2}{2} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!}$$

They are the same! This isn't a proof, but this calculation hints to the proof of Euler's eqn.

Ex: Find the degree 3 Taylor Poly. of $f(x) = 3x^3 - x^2 + 1$ near $a = -1$.

Sol: Use the formula $P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3$

$$= f(-1) + f'(-1)(x-(-1)) + \frac{f''(-1)}{2}(x-(-1))^2 + \frac{f^{(3)}(-1)}{3!}(x-(-1))^3$$

$$= f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \frac{f^{(3)}}{3!}(x+1)^3.$$

$f(x) = 3x^3 - x^2 + 1$

$f'(x) = 9x^2 - 2x$

$f''(x) = 18x - 2$

$f^{(3)}(x) = 18$

$\Rightarrow$

$f(-1) = 3(-1)^3 - (-1)^2 + 1 = -3 - 1 + 1 = -3$

$f'(-1) = 9(-1)^2 - 2(-1) = 9 + 2 = 11$

$f''(-1) = 18(-1) - 2 = -20$

$f^{(3)}(-1) = 18$

So $P_3(x) = -3 + 11(x+1) - \frac{20}{2}(x+1)^2 + \frac{18}{3!}(x+1)^3$

$$\boxed{P_3(x) = -3 + 11(x+1) - 10(x+1)^2 + 3(x+1)^3}$$

Fact: Taylor polys. approximate f. Since f is already a deg. 3 poly, $P_3(x)$ must be f itself. If you expand

$-3 + 11(x+1) - 10(x+1)^2 + 3(x+1)^3$ you will get $3x^3 - x^2 + 1$.