Longo: Math 10B - Winter 2017
Lecture Notes

Date: January 13, 2017

Section:

- [ ] §5.3

Topics Covered:

- [ ] The Fundamental Theorem of Calculus and applications

§ 5.3: The Fundamental Theorem of Calculus:

Last time we defined $\int_a^b f(x)\,dx$ as a limit of Riemann Sums:

(see past lecture notes for notation)

$$\int_a^b f(x)\,dx = \lim_{N\to\infty}\left(\sum_{i=1}^{N} f(x_{r_i})\Delta x\right)$$
$$= \lim_{N\to\infty}\left(\sum_{i=1}^{N} f(x_{\ell_i})\Delta x\right)$$

Here, we can think of "dx" as what the Δx turns into when we take limits. Since Δx gets smaller and smaller as $N\to\infty$, we think of dx as an infinitesimally small quantity. $\int$ is a big 'S', which indicates the integral comes from a sum of terms of the form $\underline{f(x_*)\Delta x}$ where Δx is meant to be an arbitrarily small quantity.

Remark: Notice the similarity with the derivative: $\frac{df}{dx} = \lim_{\Delta x\to 0} \frac{\Delta f}{\Delta x}$

Therefore, if f and x have some type of physical interpretation, the units of $\int_a^b f(x)\,dx$ are

$$(\text{units of } f) \times (\text{units of } x).$$

__Example:__ On the first day we saw that the area under the velocity curve gave us distance travelled (assume velocity is positive for simplicity). Therefore distance travelled from say $t=a$ to $t=b$ is exactly $\int_a^b v(t)\,dt$.

If velocity is measured in miles per hour, and t is measured in miles, then units of

distance traveled $= \int_a^b v(t)\,dt$ are

$$(\text{units of } v)\times(\text{units of } t) = (\text{miles}/\text{hr})\times(\text{hr}) = \boxed{\text{miles}}$$

which makes sense.

The First Fundamental Theorem of Calculus:

Sticking with the velocity example, if $r(t)$ is the function that tells us the position of an object at time t, then $r'(t)=v(t)$ is the velocity of the object (Look at you notes from last quarter!).

The total displacement from time $t=a$ to $t=b$ is equal to

$$\int_a^b v(t)\,dt$$

On the other hand, total displacement = change in position = (final position) - (initial position)

$= r(b)-r(a)$

Altogether, we have

$$r(b)-r(a) = \int_a^b r'(t)\,dt.$$

This illustrates the

1st Fundamental Theorem of Calculus: If f is a continuous fcn on the interval $[a,b]$, and if F is a fcn such that $F'(x)=f(x)$, then

$$\int_a^b f(x)\,dx = F(b)-F(a)$$

With the above notation, we call F an antiderivative of f.

Remark: ① This theorem is beautiful.

② The FTC gives us a very easy way to calculate integrals if we have an antiderivative.

③ Finding "usable" antiderivative can be extremely difficult... much harder than calculating derivatives.

Example: Suppose a ball falls of of a 5 ft. tall table. The ball's velocity after t seconds is given by $v(t) = -16t$ ft./s. How high above the ground is the ball after $\frac{1}{2}$ seconds?

Sol: Let $r(t)$ be the position of the ball after t seconds. Then:

① $r(0) = 5$.

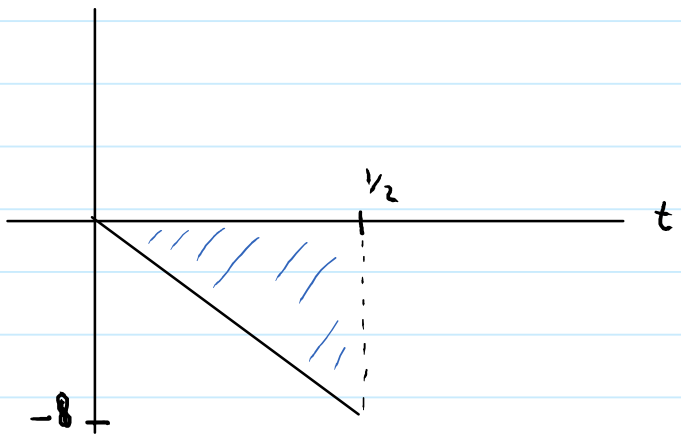
② $r'(t) = v(t)$.

③ the number we are looking for is $r(\frac{1}{2})$.

The 1st fundamental thm tells $r(\frac{1}{2}) - r(0) = \int_0^{\frac{1}{2}} v(t)\, dt = \int_0^{\frac{1}{2}} -16t\, dt$.

And therefore $\boxed{r(\frac{1}{2}) = 5 + \int_0^{\frac{1}{2}} -16t\, dt}$.

Let's calculate $\int_0^{\frac{1}{2}} -16t\, dt$ by looking at the graph of $v(t) = -16t$. from $t=0$ to $t=\frac{1}{2}$.



The graph is a line through the origin with slope -16. Since the blue triangle has area $\frac{1}{2}bh = \frac{1}{2}(\frac{1}{2})(8) = 2$

$\int_0^{\frac{1}{2}} -16t\,dt = -2.$ Notice the answer is negative because the triangle is below the t-axis.

So $\boxed{r(\tfrac{1}{2}) = 5 + \int_0^{1/2} -16t\,dt = 5 - 2 = 3}$

Remark: A better way to solve this is by finding a suitable antiderivative. We will do this later.

Example: Use the FTC to evaluate
$$\int_0^4 e^x\,dx$$

$\boxed{\text{Sol:}}$ by the FTC, $\int_0^4 e^x\,dx = F(4) - F(0)$ where $F(x)$ is a fcn whose derivative is equal to the integrand, e^x. Well,

$F(x) = e^x$ works since $F'(x) = \frac{d}{dx}(e^x) = e^x$.

Therefore
$$\int_0^4 e^x\,dx = F(4) - F(0) = e^4 - e^0 = \boxed{e^4 - 1}$$

Example: ① Use the FTC to calculate $\int_0^8 2x\,dx$.

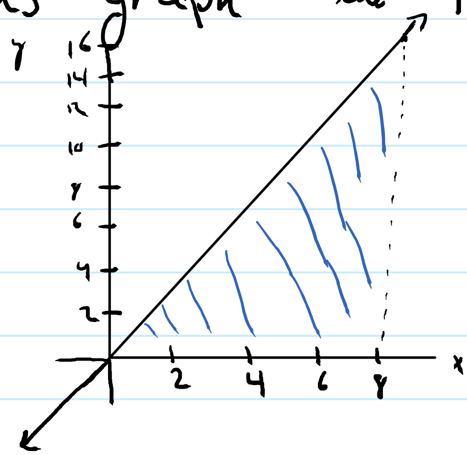
② check the answer by computing areas.

$\boxed{\text{Sol:}}$ ① By the FTC, we have
$$\int_0^8 2x\,dx = F(8) - F(0) \text{ where}$$
$F(x)$ is a fcn such that $F'(x) = 2x$. Well,

$F(x) = x^2$ works since $\frac{d}{dx}(x^2) = 2x$.

Therefore, $\int_0^8 2x\,dx = F(8) - F(0) = 8^2 - 0^2 = \boxed{64}$

(2) Lets graph the fcn $f(x) = 2x$.



Then $\int_0^8 2x\,dx$ is the area between the line and the x-axis from $x=0$ to $x=8$.

Since this area is a triangle, we have $\int_0^8 2x\,dx$ = area of Δ

$$= \tfrac{1}{2} bh$$
$$= \tfrac{1}{2}(8)(16)$$
$$= 8 \cdot 8$$
$$= \boxed{64}$$

As a final remark, we mention that the reason why the FTC is true, is basically what we did in the first lecture. If we want to calculate total change in F, i.e., $F(b) - F(a)$, we can split the interval into N-many small subintervals and estimate the change in F over the small intervals.

On the i^{th} subinterval,

$$\frac{\Delta F}{\Delta x} \approx \text{instantaneous rate of change}$$
$$= F'(x_i). \quad \text{where } x_i \text{ is some point}$$

in the i^{th} subinterval.

So $\quad \Delta F \approx F'(x_i)\Delta x.$

Note: This approximation is true if Δx is small.

Adding up over all the subintervals we have

$$F(b) - F(a) \approx \sum_{i=1}^{N} F'(x_i)\Delta x$$

The right hand side is a Riemann Sum!

Taking limits, we have

$$F(b) - F(a) = \lim_{N \to \infty} \left(\sum_{i=1}^{N} f(x_i) \Delta x \right)$$

$$= \int_a^b f(x)\,dx$$

as required.