Longo: Math 10B - Winter 2017
Lecture Notes

Date: January 18, 2017

Section:

- [ ] §5.4

Topics Covered:

- [ ] Basic properties of the definite integral: changing the bounds and "linearity" of the definite integral

§ 5.4: Properties of definite integrals:

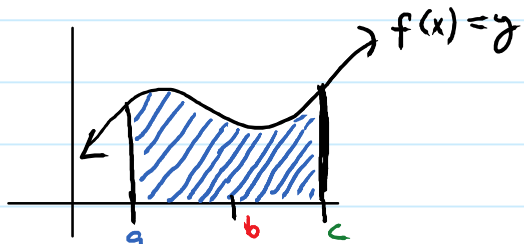
Today we will talk about properties of the definite integral that you will (hopefully) use freely by the end of the quarter. For this lecture, a, b, c are reals and f, g are cont. fcns.

Messing with the limits and integrals:

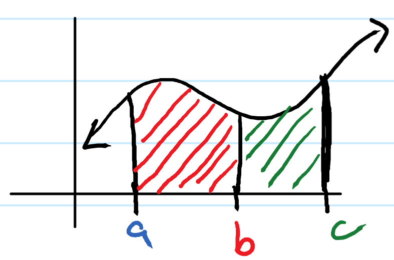
① $$\boxed{\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.}$$

Why? Suppose $a < b < C$,

Then $\int_a^c f(x)\,dx$ is the blue area



and $\int_a^b f(x)dx + \int_b^c f(x)dx$ is (red area) + (green area)



As we can see, they are the same.

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② $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

Why? Again, let's say $a < b$.

We know $\int_b^a f(x)\,dx = \lim_{N \to \infty} \left(\sum_{i=1}^{N} f(x_i)\Delta x \right)$

But now, Δx = change in x is negative!

$$\Delta x = \frac{(a-b)}{N} < 0 \qquad (a < b\,!)$$

but $\Delta x = -\frac{(b-a)}{N}$.

So
$$\int_b^a f(x)\,dx = \lim_{N\to\infty}\left(\sum_{i=1}^N f(x_i)\left(-\frac{(b-a)}{N}\right)\right)$$
$$= -\lim_{N\to\infty}\underbrace{\left(\sum_{i=1}^N f(x_i)\frac{(b-a)}{N}\right)}_{\text{Right-hand Riemann Sum for } \int_a^b f(x)dx}$$
$$= -\int_a^b f(x)\,dx$$

Intuitively: for $\int_b^a f(x)\,dx$, we are moving in the opposite direction as in $\int_a^b f(x)dx$ so $\int_b^a f(x)dx = -\int_a^b f(x)dx$

③ $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)dx \pm \int_a^b g(x)\,dx$. and

$\int_a^b c f(x)\,dx = c\int_a^b f(x)dx$

Why: Because and
$$\boxed{\begin{gathered}\sum_{i=1}^N (f(x_i)+g(x_i))\Delta x = \left(\sum_{i=1}^N f(x_i)\Delta x\right) + \left(\sum_{i=1}^N g(x_i)\Delta x\right)\\ \sum_{i=1}^N c f(x_i)\Delta x = c\sum_{i=1}^N f(x_i)\Delta x.\end{gathered}}$$.

(Then plug into def of derivative)

Example: If $\int_1^9 f(x)dx = 4$, $\int_1^4 g(x)dx = 8$ $\int_4^9 g(x)\,dx = -4$.

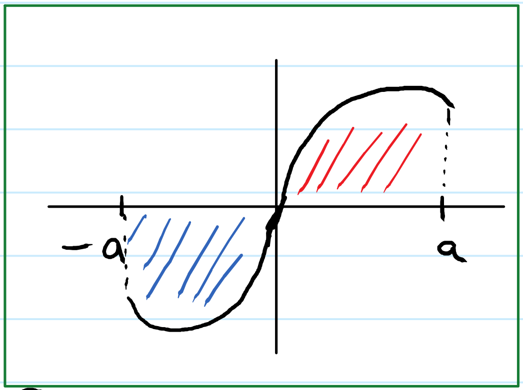
① $\int_1^9 g(x)dx = \int_1^4 g(x)dx + \int_4^9 g(x)dx = 8 + (-1) = \boxed{7}$

② $\int_1^9 f(x) + 3g(x)dx = \int_1^9 f(x)dx + 3\int_1^9 g(x)$
$= 4 + 3(7)$
$= \boxed{25.}$

Symmetry:

Recall: ① A function is called even if the graph is symmetric about the y-axis.
② A function is called odd if the graph has 180° rotational symmetry about the origin

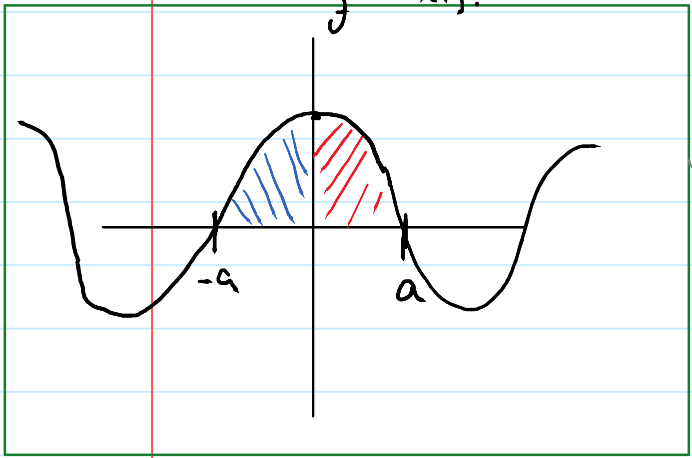
① If $f(x)$ is odd, then the graph is symmetric about the origin.



The blue area is equal to red area, so they cancel each other out.

So $\boxed{\int_{-a}^{a} f(x)\,dx = 0.}$

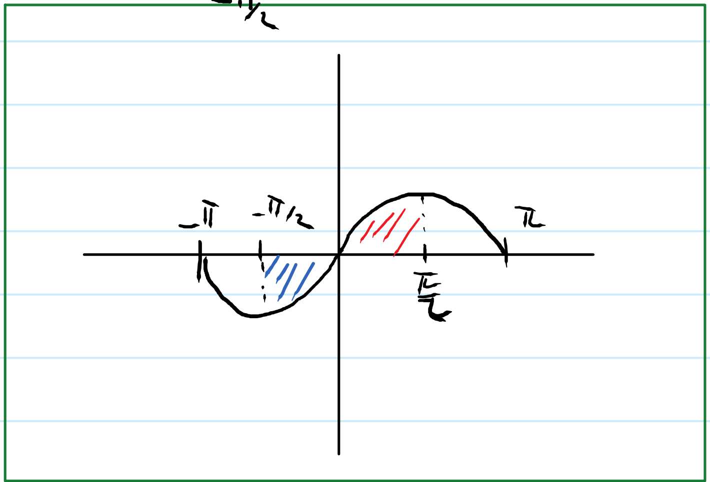
② If $f(x)$ is even, the graph is symmetric about the y-axis:



The blue area = red area

So $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$

Example: $\int_{-\pi/2}^{\pi/2} \sin(x)\,dx = 0$, $\int_{-10}^{10} x\,dx = 0$ $\int_{-10}^{10} x^3\,dx = 0$

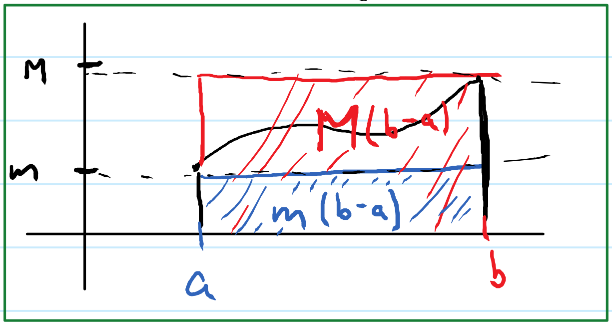


Crude Estimates:

If $m \leq f(x) \leq M$ for all $a \leq x \leq b$

Then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

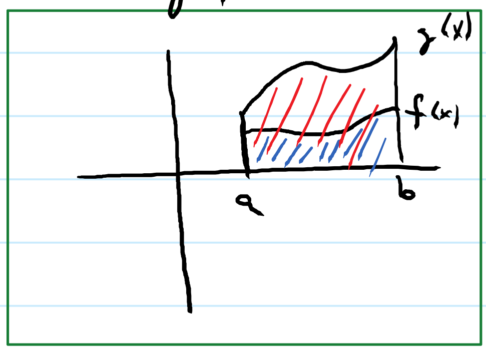
Why?



The area under the curve is $\le$ red area and $\ge$ blue area.

② Similarly, if $f(x) \le g(x)$ for all $a \le x \le b$. Then the graph of $g(x)$ is above the graph of f on this interval

So $\boxed{\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx}$



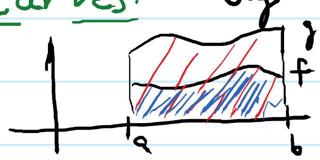
Example: Since $0 \le \cos\theta \le 1$ for $0 \le \theta \le \frac{\pi}{2}$,

$$0\left(\frac{\pi}{2}\right) \le \int_0^{\frac{\pi}{2}} \cos(\theta)\,d\theta \le 1\left(\frac{\pi}{2}\right)$$

$$\boxed{0 \le \int_0^{\frac{\pi}{2}} \cos(\theta)\,d\theta \le \frac{\pi}{2}}$$

Rmk: We will later that $\int_0^{\frac{\pi}{2}} \cos(\theta)\,d\theta = 1$.

Area between Curves: Say $f(x) \le g(x)$ from a to b.



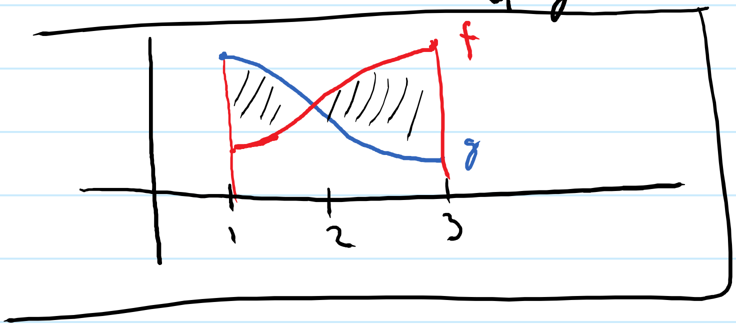
The area between the graphs of f, g is (the red area) $-$ (the blue area)

The area between the graphs is $\int_a^b g(x) - f(x)\,dx$

Example: Suppose $f(x) \le g(x)$ for $1 \le x \le 2$ and $g(x) \le f(x)$ for $2 \le x \le 3$.

Suppose $\int_1^2 f(x)\,dx = 2$ $\int_2^3 f(x)\,dx = 4$

$\int_1^2 g(x)\,dx = 7$ $\int_2^3 g(x)\,dx = 2$.



Then the area between the graphs of f & g is:

$$\boxed{\int_1^2 (g(x) - f(x))\,dx + \int_2^3 (f(x) - g(x))\,dx}$$

$$= \int_1^2 g(x)\,dx - \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx - \int_2^3 g(x)\,dx$$

$$= 7 - 2 + 4 - 2$$

$$\boxed{= 7}$$

Finally, we come to the (very important:) Mean Value thm:

Thm: If f is continuous from a to b. Then the average (mean) value of f is

$$\boxed{\frac{1}{b-a} \int_a^b f(x)\,dx}$$

Why? Intuitive Answer: To find the average of a finite set of numbers, you add them all up and then divide by however many numbers you have. Think of $\int_a^b f(x)\,dx$ as "the sum of all possible $f(x)$ values, and think of $b-a$ as how many numbers you have.

Remember: $b-a$ is the length of the interval from a to b.

Example: Suppose a cars speed, in mph, at time t is given by $v(t) = 2t$. What is the car's average speed after four hours?

Sol: The car's average speed from 0 to 4 hours is

$$\frac{1}{4-0}\int_0^4 2t\,dt.$$

From lecture 1, $\int_0^4 2t\,dt = 16$. So the average speed is $\frac{1}{4}(16) = 4$.

Remark: This is not a surprise! $\int_0^4 2t\,dt =$ distance travelled after 4 hours. If the car goes 16 miles over 4 hours, its average speed is 4 mph!

Remark: I just noticed how slow this car is going... oops.