Longo: Math 10B - Winter 2017
Lecture Notes

Date: January 20, 2017

Section:

- [ ] §6.1
- [ ] §6.2

Topics Covered:

- [ ] Antiderivatives from graphs
- [ ] Constructing antiderivatives

§6.1: Antiderivatives graphically and numerically:

Recall the (first) fundamental thm of calculus:

> If f is a cont. fcn on $[a,b]$, and if $F'(x)=f(x)$, then
> $$\int_a^b f(x)\,dx = F(b)-F(a).$$

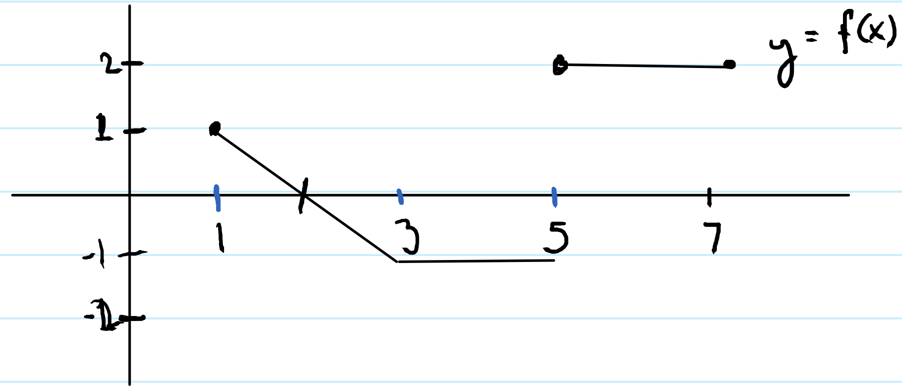
It should be clear that if we are given a fcn f, it is crucial that we can find a function $F(x)$ where $F'(x)=f(x)$. We call F an antiderivative of f.

> **Note:** There are infinitely many antiderivatives for any given function. For example
> $$F_1(x)=x^2, \quad F_2(x)=x^2+1, \quad F_3(x)=x^2-9$$
> are all antiderivatives for $f(x)=2x$. In fact, if $F(x)$ is an antiderivative of $f(x)=2x$, then $F(x)=x^2+C$ for some constant C.

Graphically Constructing antiderivatives:

Last quarter, we constructed the graph of $f'(x)=y$ if we already know the shape of the graph of $f(x)=y$. In this class we want to construct the graph of an antiderivative of f.

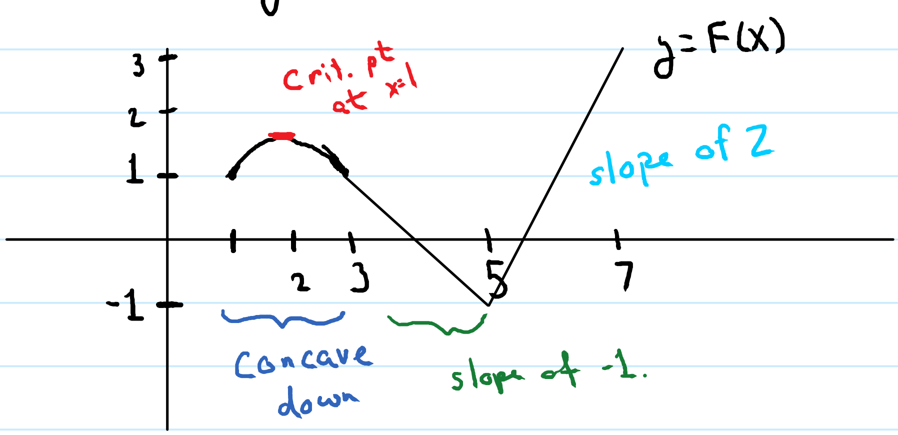
Example: If f is the function whose graph is given below, Sketch a graph of an antiderivative F of f that satisfies $F(1)=-1$



Sol: Let's extract as much information as possible.
If F is an antiderivative, then:

1. The slope of the graph of F starts at 1 when $x=1$ and decreases to -1 by the time $x=3$. In particular, F is concave down on $[1,3]$.
2. As $f(2)=0$, F has a critical pt. at 2
3. The slope is a flat -1 from 3 to 5.
4. The slope jumps to 2 from 5 to 7.

We still can't graph unless we know where to start. Since we are given that $F(1)=1$, we start there.



How did I get those specific values for $F(x)$?
By
Constructing the antiderivative numerically:

Exercise: Using the example above, determine $F(2)$, $F(3)$, $F(5)$.

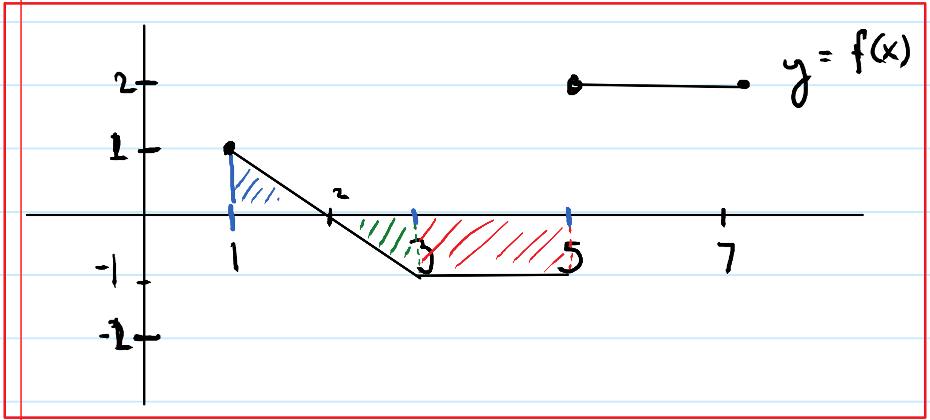
Remember that: ① $F(1)=1$ ⟵ Given

② $\int_1^c f(x)\,dx = F(c)-F(1)$ ⟵ FTC.

So for any number c, $F(c)=\int_1^c f(x)\,dx + F(1) \Rightarrow$

$F(c) = \int_1^c f(x)\,dx + 1$.

So we just need to calculate integrals (using area).



Notice, $\int_1^2 f(x)\,dx = (\text{blue area})$

$= \frac{1}{2}(1)(1)$

area of triangle.

So $F(2)=\int_1^2 f(x)\,dx + 1 = \frac{1}{2}+1$

$\boxed{=\frac{3}{2}}$

$\int_1^3 f(x)\,dx = (\text{blue area}) - (\text{green area}) = \frac{1}{2} - \frac{1}{2} = 0,$

So $F(3) = \int_1^3 f(x)\,dx + 1 = 0+1 = \boxed{1}$

$\int_1^5 f(x)\,dx = (\text{blue area}) - (\text{green area}) - (\text{red area}) = (\frac{1}{2}) - (\frac{1}{2}) - (2)(1)$

$= -2$

So $F(5) = \int_1^5 f(x)\,dx + 1 = -2+1 = \boxed{-1}$

Note: we could have simplified computation using the identity: $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

Conclusion: We can construct antiderivatives if we somehow know some values of the definite integral.

§6.2 - Constructing antiderivatives analytically:

First goal: Let f be a (continuous) fcn. Let's first prove that if F, G are antiderivatives of f, then $F(x) = G(x) + C$ for some C.

Notice: If F is an antiderivative of the function $f(x) = 0$, then $F(x) = C$ for some constant C.

Why? We know $F'(x) = 0$. graphically, this means the slope of the tangent line of F is always 0. Therefore, the graph of F is a horiz. line. This means $F(x)$ is a constant fcn. i.e. $F(x) = c$ for some C.

Now if $G'(x) = F'(x)$, then $(G-F)'(x) = 0$.
By the above, $G(x) - F(x) = C$ for some constant C. ✓

This means that as soon as we know one antiderivative, we know them all.

Def: If F is an antiderivative of f. Then the indefinite integral of f is:

$$\int f(x)\,dx := F(x) + C.$$

Note: Usually we just say "integrate f" instead of "Find an antiderivative of f."

Some straight forward antiderivatives.

Recall: (1) $\frac{d}{dx}(kx) = k$
(2) $\frac{d}{dx}(x^n) = n x^{n-1}$

(To get (2), the rule is to multiply by the exponent, and then reduce the exponent.)

Therefore, we can say ① $\int k\,dx = kx + C$

② $\int x^n dx = \frac{x^{n+1}}{n+1} + C.$

(To get ②, increase the exponent, and then divide by the exponent. This is the opposite of before.)

Example: ① $\int \frac{1}{4}x^3 - 2x^2 - \frac{1}{2}x\,dx$

$= \frac{1}{4}\int x^3 dx - 2\int x^2 dx - \frac{1}{2}\int x\,dx$

$= \frac{1}{4}\left(\frac{x^4}{4}\right) - 2\left(\frac{x^3}{3}\right) - \frac{1}{2}\left(\frac{x^2}{2}\right) + C$

$\boxed{= \frac{x^4}{16} - \frac{2}{3}x^3 - \frac{1}{4}x^2 + C}$

② $\int \frac{3}{x^4}dx = \int 3x^{-4}dx$

$= 3\left(\frac{x^{-3}}{-3}\right) + C$

$= \boxed{-x^{-3} + C}$

③ $\int \frac{2x^2\sqrt{x} + 9}{x^2}dx = \int \frac{2x^2\sqrt{x}}{x^2} + \frac{9}{x^2}dx$

$= \int 2\sqrt{x}\,dx + \int \frac{9}{x^2}dx$

$= \int 2x^{\frac{1}{2}} + 9x^{-2}dx$

$= \frac{2x^{\frac{1}{2}+1}}{\left(\frac{3}{2}\right)} + 9\left(\frac{x^{-1}}{-1}\right) + C$

$= \left(\frac{2}{3}\right)\left(2x^{3/2}\right) - 9x^{-1} + C$

$= \frac{4}{3}x^{3/2} - \frac{9}{x} + C$

Example: Calculate $\int_{-3}^{3} 2x^3 dx.$

Notation: Instead of $\int_a^b f(x)dx = F(b) - F(a)$, we usually write $\int_a^b f(x)dx = (F(x))\Big|_a^b$ "F evaluated from a to b".

$$\int_{-3}^{3} 2x^3\,dx = 2\left(\frac{x^4}{4}\right)\Big|_{-3}^{3} = \frac{x^4}{2}\Big|_{-3}^{3} = \frac{3^4}{2} - \frac{(-3)^4}{2}$$

$$= \frac{81}{2} - \frac{81}{2}$$

$$= \boxed{0}$$

$\boxed{Q?}$ Was there a better way to do this?

$\boxed{\text{Ans.}}$ Yes, $f(x) = 2x^3$ is an odd fcn, so

$$\int_{-3}^{3} 2x^3\,dx = 0$$