Longo: Math 10B - Winter 2017
Lecture Notes

Date: January 23, 2017

Section:

- §6.2

Topics Covered:

- More basic antiderivatives (the calm before the storm)

More basic antiderivatives:

Last quarter, we saw:

Proposition: ① $\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(a^x) = \ln(a)a^x$

② $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ if $x > 0$.

If $x < 0$, then $\frac{d}{dx}(\ln(-x)) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$.

③ $\frac{d}{dx}(\sin(x)) = \cos(x)$

$\frac{d}{dx}(\cos(x)) = -\sin(x)$

$\frac{d}{dx}(\tan(x)) = \frac{1}{\cos^2(x)} = \sec^2(x)$

Using this information, we can come up with some antiderivatives:

Proposition: ① $\boxed{\int e^x\,dx = e^x + C}$

$$\boxed{\int a^x\,dx = \frac{a^x}{\ln(a)} + C}$$

② If $x > 0$, $\int \frac{1}{x}\,dx = \ln(x) + C$

If $x < 0$, $\int \frac{1}{x}\,dx = \ln(-x) + C$.

we can write this more simply as

$$\boxed{\int \frac{1}{x}\,dx = \ln(|x|) + C}$$

③ $\boxed{\int \cos(x)\,dx = \sin(x) + C}$

$$\boxed{\int \sin(x)\,dx = -\cos(x) + C}$$

$$\boxed{\int \frac{1}{\cos^2(x)}\,dx = \tan(x) + C.}$$

Reminder: ① Do not forget the C!

② You can always double check your indefinite integral by differentiating!

For example, lets check $\int a^x dx = \frac{a^x}{\ln(a)} + C$.

$$\frac{d}{dx}\left(\frac{a^x}{\ln(a)} + c\right) = \frac{d}{dx}\left(\frac{a^x}{\ln(a)}\right) + \frac{d}{dx}(c)$$
$$= \frac{1}{\ln(a)} \frac{d}{dx}(a^x) + 0$$
$$= \frac{1}{\cancel{\ln(a)}}\left(\cancel{\ln(a)}\, a^x\right)$$
$$a^x \checkmark$$

Examples: ① $\int 3e^x + 2\,dx = \int 3e^x dx + \int 2\,dx$
$$= 3\int e^x dx + \int 2\,dx$$
$$= \boxed{3e^x + 2x + C}$$

② $\int \frac{x^2+2}{x} dx = \int \frac{x^2}{x} + \frac{2}{x} dx$
$$= \int x\,dx + \int \frac{2}{x} dx$$
$$= \int x\,dx + 2\int \frac{1}{x} dx$$
$$= \boxed{\frac{x^2}{2} + 2\ln(|x|) + C}$$

③ $\int \frac{\cos(x)}{2} + 4\sin(x)\,dx = \frac{1}{2}\int \cos(x)dx + 4\int \sin(x)dx$
$$= \frac{1}{2}\sin(x) + 4(-\cos(x) + C$$
$$= \boxed{\frac{1}{2}\sin(x) - 4\cos(x) + C}$$

④ $\int (2x+3)^2 dx = \int (2x+3)(2x+3)\,dx$
$$= \int 4x^2 + 12x + 9\,dx$$
$$= 4\int x^2 dx + 12\int x\,dx + \int 9\,dx$$
$$= 4\left(\frac{x^3}{3}\right) + 12\left(\frac{x^2}{2}\right) + 9x + C$$
$$= \boxed{\frac{4}{3}x^3 + 6x^2 + 9x + C}$$

!Warning! $\int (2x+3)^2 dx$ is <u>not</u> $\frac{(2x+3)^3}{3} + C$.

Why? Let's check: $\frac{d}{dx}\left(\frac{(2x+3)^3}{3}\right) = 3(2x+3)^2(2)$
$$= 6(2x+3)^2 \ldots$$

The problem is that we had to use the chain rule! Later, we will talk about something that is like the "anti-chain rule".

Examples: ① Find an antiderivative, F, of $f(x) = 3\cdot(5^x)$ such that $F(1) = 4$.

Sol: Let's find: $\int 3\cdot 5^x\,dx = 3\int 5^x\,dx$

$$= 3\int 5^x\,dx$$

$$= \frac{3}{\ln(5)}\cdot 5^x + C.$$

So $F(x)$ has the form

$F(x) = \frac{3}{\ln(5)}\cdot 5^x + C$, for some constant C. To find C, remember that we have $F(1) = 4$.

$$\Rightarrow \underbrace{\frac{3}{\ln(5)}\cdot 5^{(1)} + C}_{F(1).} = 4$$

$$\Rightarrow C = 4 - 5\cdot\left(\frac{3}{\ln(5)}\right)$$

$$\Rightarrow C = 4 - \frac{15}{\ln(5)}$$

All together: $\boxed{F(x) = \frac{3}{\ln(5)}\cdot 5^x + \left(4 - \frac{15}{\ln(5)}\right).}$

② Evaluate: $\int_1^e \frac{1}{2x}\,dx = \frac{1}{2}\int_1^e \frac{1}{x}\,dx$

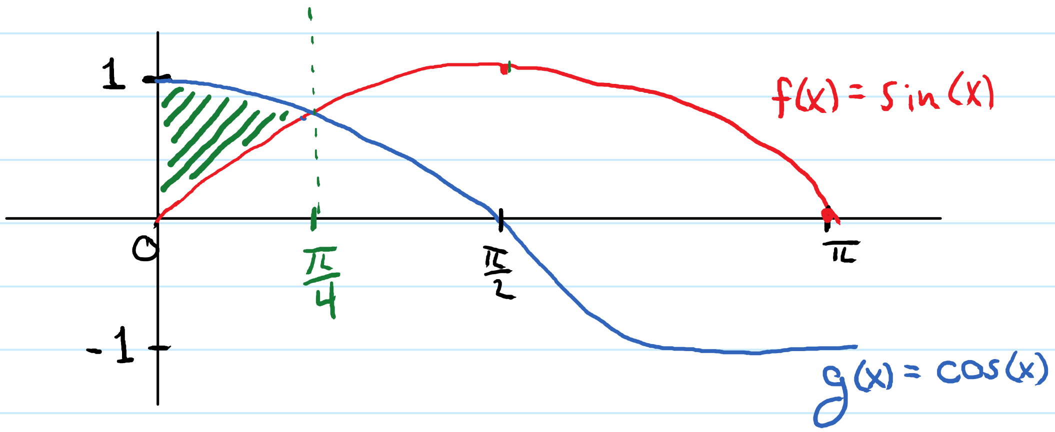
$$= \frac{1}{2}\left(\ln(|x|)\right)\Big|_1^e$$

Recall, this means "evaluate the fcn from 1 to e," which means "$F(e) - F(1)$" where F is the antiderivative.

$$\frac{1}{2}\left(\ln(|x|)\right)\Big|_1^e = \frac{1}{2}\ln(e) - \frac{1}{2}\ln(1)$$
$$= \frac{1}{2}(1) - \frac{1}{2}(0)$$
$$= \boxed{\frac{1}{2}}$$

③ Calculate the area between the graphs of $f(x) = \sin(x)$ and $g(x) = \cos(x)$ where $0 \le x \le \frac{\pi}{4}$

Sol: Let's first draw the picture:



Note: From 0 to $\frac{\pi}{4}$, $\sin(x) \le \cos(x)$, and at $\frac{\pi}{4}$, $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

So the area we want is: (See Lecture 5).

$$\int_0^{\frac{\pi}{4}} \cos(x) - \sin(x)\,dx = \int_0^{\frac{\pi}{4}} \cos(x)\,dx - \int_0^{\frac{\pi}{4}} \sin(x)\,dx$$
$$= \sin(x)\Big|_0^{\frac{\pi}{4}} - \left(-\cos(x)\right)\Big|_0^{\frac{\pi}{4}}$$
$$= \sin(x)\Big|_0^{\frac{\pi}{4}} + \cos(x)\Big|_0^{\frac{\pi}{4}}$$
$$= \left(\sin\left(\frac{\pi}{4}\right) - \sin(0)\right) + \left(\cos\left(\frac{\pi}{4}\right) - \cos(0)\right)$$
$$= \left(\frac{\sqrt{2}}{2} - 0\right) + \left(\frac{\sqrt{2}}{2} - 1\right)$$
$$= \boxed{\sqrt{2} - 1}$$

Additional Practice:

(1) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{3}{\cos^2(x)}\,dx = 3\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{\cos^2(x)}\,dx$

$$= 3\tan(x)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= 3\tan\left(\frac{\pi}{4}\right) - 3\tan\left(-\frac{\pi}{4}\right)$$

$$= 3\cdot(1) - 3(-1)$$

$$= \boxed{6}$$

(2) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2(x)\,dx$.

Remark: This is not one of the "standard" forms, so we should manipulate the integrand first.

$$\int_0^{\frac{\pi}{4}} \tan^2(x)\,dx = \int_0^{\frac{\pi}{4}} \frac{\sin^2(x)}{\cos^2(x)}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\cos^2(x)}{\cos^2(x)}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2(x)} - \frac{\cos^2(x)}{\cos^2(x)}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2(x)}\,dx - \int_0^{\frac{\pi}{4}} 1\,dx$$

$$= \tan(x)\Big|_0^{\pi/4} - x\Big|_0^{\pi/4}$$

$$= \left(\tan\left(\frac{\pi}{4}\right) - \tan(0)\right) - \left(\frac{\pi}{4} - 0\right)$$

$$= (1 - 0) - \frac{\pi}{4}$$

$$= \boxed{1 - \frac{\pi}{4}}$$

(3) Find an antiderivative, F, of $f(x) = \dfrac{\sqrt{x} - x^3}{x^4}$

that satisfies $F(1) = 3$. (Assume domain of f is $(0, \infty)$)

Sol: First find the indefinite integral of f.

$$\int \frac{\sqrt{x} - x^3}{x^4}\,dx = \int \frac{\sqrt{x}}{x^4}\,dx - \int \frac{x^3}{x^4}\,dx$$

$$= \int \frac{x^{1/2}}{x^4}\,dx - \int \frac{1}{x}\,dx$$

$$= \int x^{\frac{1}{2}-4}\,dx - \int \frac{1}{x}\,dx$$

$$= \int x^{-\frac{7}{2}}\,dx - \int \frac{1}{x}\,dx$$

$$= \frac{x^{-\frac{7}{2}+1}}{\left(\frac{-7}{2}+1\right)} - \ln|x| + C$$

$$= \frac{x^{-\frac{5}{2}}}{-\frac{5}{2}} - \ln|x| + C$$

$$= -\frac{2}{5}x^{-\frac{5}{2}} - \ln|x| + C.$$

So F has the form $F(x) = -\frac{2}{5}x^{-\frac{5}{2}} - \ln|x| + C.$

To find C, we have: $F(1) = 3 \Rightarrow$

$$\underbrace{-\frac{2}{5}(1)^{-\frac{5}{2}} - \ln(1) + C}_{F(1)} = 3$$

$$\Rightarrow \quad -\frac{2}{5} - 0 + C = 3$$

$$\Rightarrow \quad C = 3 + \frac{2}{5}$$

$$C = \frac{17}{5}$$

$$\Rightarrow \quad \boxed{F(x) = -\frac{2}{5}x^{-\frac{5}{2}} - \ln x + \frac{17}{5}}$$

Hopefully this is enough examples.