Longo: Math 10B - Winter 2017
Lecture Notes

Date: January 25, 2017

Section:
- [ ] §6.3
- [ ] §6.4

Topics Covered:
- [ ] Introduction to differential equations and elementary equations of motion.
- [ ] The 2nd Fundamental Theorem of Calculus

§6.3: Differential equations and equations of motion:

Definition: ① A differential equation is an equation of the form:

$$\boxed{\frac{dy}{dx} = f(x)}$$

② A general solution of a differential equation, is the general antiderivative

$$\boxed{y = F(x) + C}$$

where $F'(x) = f(x)$.

Warning: Since there are many antiderivatives, the general solution is not a single graph. Instead it is a family of graphs indexed by C.

This definition is easier to understand via examples.

Example: Find and graph the general solution to

$$\frac{dy}{dx} = x + 1.$$

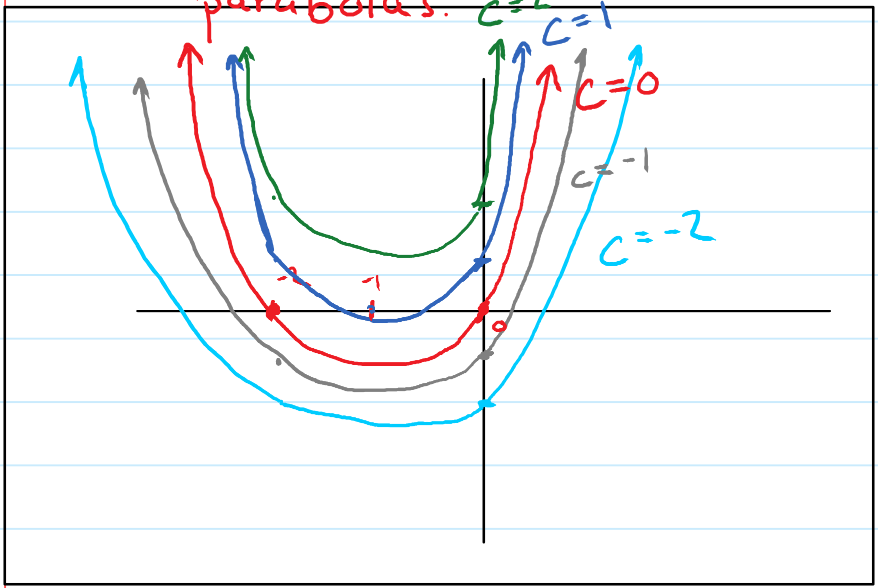
Sol: We must find a general antiderivative of the function $f(x) = x + 1$. We saw last lecture that any antiderivative of f has the form

$$F(x) = \frac{x^2}{2} + x + C.$$

* Solution just means any eqn where $\frac{dy}{dx} = x+1$.

So the general solution* is $y = \frac{x^2}{2} + x + C$

If we let C vary, we see that the general solution is a family of parabolas.



When $C=0$, we have $y=\frac{x^2}{2}+x$
$= x\left(\frac{x}{2}+1\right)$

which is a parabola that opens upwards and has roots at $x=0$ and $x=-2$.

If we increase C, say $C=1 \Rightarrow y=\frac{x^2}{2}+x+1$, then the graph gets shifted up or down by C.

So for example, the dark blue graph $(C=1)$ is just the red graph $(C=0)$ shifted up by 1.

If we want to specify a single curve, we need an initial value.

<u>Example</u>: (Initial value problem) Find the solution to the initial value problem

$$\frac{dy}{dx} = x+1 \quad ; \quad y(4)=6$$

<u>Sol</u>: We just found the general solution

$$y=\frac{x^2}{2}+x+C$$

Since we know $y(4)=6$, (i.e. when $x=4, y=6$)

we have:
$$6=\frac{(4)^2}{2}+4+C$$
$$\Rightarrow 6=8+4+C$$
$$\Rightarrow C=-6$$

So the solution to the IVP is $\boxed{y=\frac{x^2}{2}+x-6}$.

Application to equations of motion:

Setup: We want to find equations that describe the motion of an object falling freely (near the Earth's surface for example).

Fact: If an object is in free fall, it's acceleration due to gravity is constant. If $s(t)$ is the position function for the object. Then

$$\frac{ds}{dt} = v(t) \text{ is velocity and}$$
$$\frac{d^2s}{dt^2} = v'(t) = a(t) \text{ is acceleration.}$$

Since acceleration is constant, we say

$$\frac{d^2s}{dt^2} = \frac{dv}{dt} = -g \qquad (g \text{ stands for gravity})$$

On Earth, $g \approx 9.8 \, m/s^2$.

Example: A ball is thrown upwards with a speed of $15 \, m/s$ from a height of $2.5 m$.

① Find formulae for the ball's position ($s(t)$) and velocity ($v(t)$).

② What is the speed of the ball when it hits the ground?

Sol: We know the ball's initial position is $2.5 m$ and initial velocity is $15 \, m/s$. So we have

$$s(0) = 2.5 \quad \text{and} \quad v(0) = 15.$$

we also know: (*) $\frac{dv}{dt} = -9.8 \, m/s^2$ by the discussion above. The differential eqn (*) tells us

$$v(t) = -9.8t + v_0 \quad \text{for some}$$

constant v_0. Since $v(0) = 15$ m/s, we get

$$15 = \underbrace{-9.8(0) + v_0}_{v(0)}$$

$\Rightarrow$ $\boxed{v_0 = 15}$ and therefore $v(t) = -9.8t + 15$.

Now we have the initial value problem:

$$v(t) = \frac{ds}{dt} = -9.8t + 15 \quad \text{and} \quad s(0) = 2.5.$$

By computing an antiderivative, we see

$$s(t) = -\frac{9.8t^2}{2} + 15t + s_0$$ for some

constant s_0. Since $s(0) = 2.5$, we have

$$2.5 = -\frac{9.8(0)^2}{2} + \cancel{15(0)} + s_0 \Rightarrow s_0 = 2.5$$

Therefore

$$s(t) = \frac{-9.8}{2}t^2 + 15t + 2.5$$

② The ball hits the ground when it's height is zero, I.e., when $s(t) = 0$.

Let's solve:

$$s(t) = 0 \Rightarrow$$

$$0 = -4.9t^2 + 15t + 2.5$$

Quadratic formula $\Rightarrow$

$$t = \frac{-15 \pm \sqrt{(15)^2 - 4(-4.9)(2.5)}}{2(-4.9)}$$

Calculator $\approx$ $\cancel{-1.6}$ or $\boxed{3.2}$ s.

Time can't be negative.

The velo. at $t = 3.2$ is $v(3.2) = (-9.8)(3.2) + 15 = -16.36 \frac{m}{s}$

Remark: If ① g = accelleration due to gravity
② v_0 = initial velocity
= velocity at time $t=0$
③ s_0 = initial height
= height at time $t=0$,
then using the calculation we just did,

$$s''(t) = v'(t) = a(t) = -g$$
$$s'(t) = v(t) = -gt + v_0$$
$$s(t) = -\frac{g}{2}t^2 + v_0 t + s_0$$

§6.4: The 2^{nd} Fundamental Thm of Calculus:

The first FTC heavily relies on the ability to find an anti-derivative. We saw last quarter that not every fcn has a derivative. This motivates the question:

Q! If f is a cont. fcn, does f have an antiderivative?

Ans: Yes, but the antiderivative might not look "nice".

Idea: By the 1^{st} FTC, if $G(x)$ satisfies, $G'(x) = f(x)$ (and f is cont. on an interval), then

$$\boxed{\int_a^b f(t)\,dt = G(b) - G(a).}$$

What if we let b vary? Replace b by the variable x, then

$$G(x) - G(a) = \int_a^x f(t)\,dt.$$

$$\Rightarrow \boxed{G(x) = \int_a^x f(t)\,dt + G(a).}$$

Summary: If $G'(x) = f(x)$, then $G(x) = \int_a^x f(t)\,dt + G(a)$

So if f is a cont. fcn on an interval, and if a is in that interval, then the fcn:

$$F(x) = \int_a^x f(t)\,dt$$

must be an antiderivative of f. This is called the 2nd Fundamental Thm of Calculus. We prove this nex time.