Longo: Math 10B - Winter 2017
Lecture Notes

Date: January 27, 2017

Section:

- [ ] §6.4

Topics Covered:

- [ ] The Second Fundamental Theorem of Calculus and applications
- [ ] The proof of the First Fundamental Theorem of Calculus

6.4 Cont.:

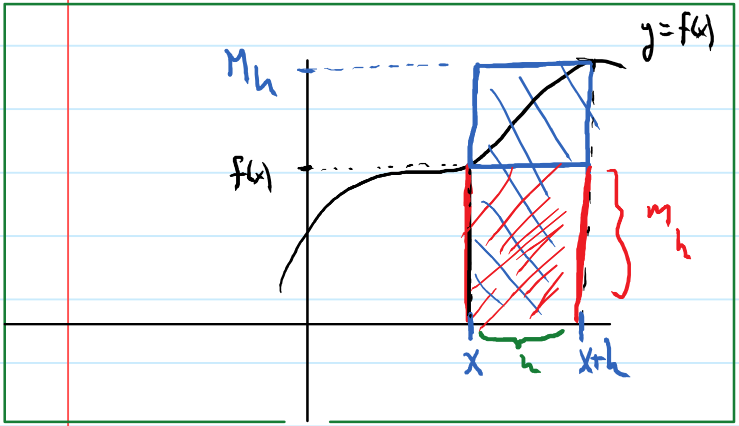
From last time:

Theorem: (2nd Fundamental Thm of Calculus)
Let f be continuous on an interval, and let a be a number in that interval. Then the fcn
$$F(x) = \int_a^x f(t)\,dt$$
satisfies $F'(x) = f(x)$.

Proof: By definition, $F'(x) = \lim_{h\to 0} \frac{1}{h}(F(x+h) - F(x))$

$$= \lim_{h\to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \right)$$

(See Lecture 5) $\Big\{$
$$= \lim_{h\to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt \right)$$
$$= \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt.$$



Let M_h be the max of f on $[x, x+h]$ and m_h be the min. of $f(x)$ on $[x, x+h]$. Then:

$$\underbrace{h \cdot m_h}_{\text{red area}} \le \int_x^{x+h} f(t)\,dt \le \underbrace{h \cdot M_h}_{\text{blue area}} \qquad \text{(see picture and lecture 5)}$$

$$\Rightarrow \qquad m_h \le \frac{1}{h} \int_x^{x+h} f(t)\,dt \le M_h.$$

Taking limits as $h \to 0$ we get

$$(**) \qquad \lim_{h\to 0} m_h \le \underbrace{\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt}_{F'(x)} \le \lim_{h\to 0} M_h.$$

Since f is continuous, m_h and $M_h \to f(x)$ as $h \to 0$.

So $(**)$ becomes $f(x) \le F'(x) \le f(x)$, which implies $F'(x) = f(x)$ ∎

Remark: ① To understand the blue box, remember since f is cont., all values of $f(c)$ are very close to $f(x)$ if c is close to x. So if we are looking at a smaller and smaller interval $([x, x+h])$ $f(c)$ is very close to $f(x)$ if c is in $[x, x+h]$. In particular, the max and min of f on $[x, x+h]$ approach $f(x)$ as $h \to 0$.

② The technique used on $(**)$ Is called the Squeeze Thm.

Let's use this to quickly prove:

Theorem: (1st Fundamental Thm of Calculus)

Let f be cont. on the interval $[a,b]$, and let $G(x)$ be ANY antiderivative of f (i.e., $G'(x) = f(x)$).
Then $\boxed{\int_a^b f(t)\,dt = G(b) - G(a)}$

Pf: Since G and $F(x) = \int_a^x f(t)\,dt$ are both antiderivatives of f, they must differ by a constant.

Say $G(x) = F(x) + C$
$= \int_a^x f(t)\,dt + C$ for some constant C.

Then $G(b) - G(a) = \left(\int_a^b f(t)\,dt + C\right) - \left(\underbrace{\int_a^a f(t)\,dt}_{=0} + C\right)$

$$= \int_a^b f(t)\,dt + C - C$$
$$= \int_a^b f(t)\,dt \quad \checkmark$$

The 2^{nd} FTC gives us a way to construct "nonelementary" antiderivatives.

"Non elementary" means it is impossible to find a formula for the fcn in terms of polys., exponentials, trig. fcns, etc.

<u>Example</u>: ① Find an antiderivative, F, of $f(x) = e^{-x^2}$ that satisfies $F(0) = 1$.

② Use Right-hand Riemann sums to estimate $F(1)$.

<u>Remark</u>: It is well known that $f(x)$ has no elementary antiderivative.

<u>Sol</u>: ① By the 2^{nd} FTC, we know that for any constant C, the fcn $F(x) = \int_0^x e^{-t^2}dt + C$ satisfies

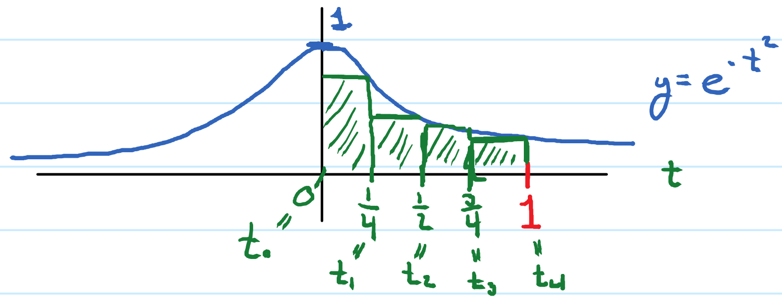
$$F'(x) = e^{-x^2}.$$

Since we want $F(0)=1$, we want to pick C so that $1 = \underbrace{\int_0^0 e^{-t^2}dt + C}_{F(0)}$. But $\int_0^0 e^{-t^2}dt = 0$

So $C = 1$.

The fcn we want is therefore $F(x) = \int_0^x e^{-t^2}dt + 1$.

② Since we wish to estimate $F(1) = \int_0^1 e^{-t^2}dt + 1$, lets use Right hand Riemann sums $(N=4)$ to estimate $\int_0^1 e^{-t^2}dt$, which is the area under the curve $y = e^{-t^2}$. Let's sketch the graph:



Recall, we can chop $[0,1]$ into four even pieces, then calculate rectangles on each subinterval where the height is the value of the fcn on the right hand side of the subinterval.

$$\int_0^1 e^{-t^2}\,dt \approx \sum_{i=1}^{4} f(t_i)\,\Delta t = f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t + f(t_4)\Delta t$$

where $f(t)=e^{-t^2}$, $\Delta t = \frac{1-0}{4} = \frac{1}{4}$.

$$\int_0^1 e^{-t^2}\,dt \approx e^{-(\frac{1}{4})^2}\left(\tfrac{1}{4}\right) + e^{-(\frac{1}{2})^2}\left(\tfrac{1}{4}\right) + e^{-(\frac{3}{4})^2}\left(\tfrac{1}{4}\right) + e^{-(1)^2}\left(\tfrac{1}{4}\right)$$

$$= \frac{1}{4}\left(e^{-\frac{1}{16}} + e^{-\frac{1}{4}} + e^{-\frac{9}{16}} + e^{-1}\right)$$

$$\approx .664$$

Calculator.

Finally, we have

$$F(1) = \int_0^1 e^{-t^2}\,dt + 1 \approx 0.664 + 1 = \boxed{1.664}$$

<u>Example</u>: (1) Let $F(x) = \int_2^x \frac{1}{\ln(t)}\,dt$. Calculate $F'(x)$.

(2) Let $G(x) = \int_2^{x^4} \frac{1}{\ln(t)}\,dt$. Calculate $G'(x)$.

Remark: The fcn $\int_0^x \frac{1}{\ln(t)}\,dt$ is called the **Logarithmic integral fcn** and is denoted $Li(x)$. An important result in number theory is that if x is large, $Li(x)$ is approximately the number of prime numbers $\leq x$.

Thanks to Corey Stone for suggesting this example.

Sol: (1) The 2^{nd} FTC tells us that $F(x) = \int_2^x \frac{1}{\ln(t)}\,dt$ is an antiderivative of the fcn $f(x) = \frac{1}{\ln(x)}$.
So we automatically get $\boxed{F'(x) = \frac{1}{\ln(x)}.}$

(2) Notice that $F(x^4) = \int_2^{x^4} \frac{1}{\ln(t)}\,dt = G(x)$.
So if $h(x) = x^4$, we have:
$G(x) = F(x^4) = F(h(x)) = (F \circ h)(x)$.
To find $G'(x)$ we use the chain Rule!

$$G'(x) = (F \circ h)'(x) = F'(h(x)) \cdot h'(x).$$

Since $F'(x) = \frac{1}{\ln(x)}$, and $h'(x) = \frac{d}{dx}(x^4) = 4x^3$, we get:

$$\boxed{\begin{aligned} G'(x) &= F'(x^4) \cdot 4x^3 \\ &= \left(\frac{1}{\ln(x^4)}\right) \cdot 4x^3 = \frac{4x^3}{\ln(x^4)} \end{aligned}}$$