Longo: Math 10B - Winter 2017
Lecture Notes

Date: January 30, 2017

Section:

- [ ] §7.1

Topics Covered:

- [ ] The Method of Substitution (The anti-chain rule)

§7.1: The Method of Substitution:

In Chapter 7 we will discuss a series of techniques to calculate antiderivatives. The first technique is the simplest, and you can think of it as the "anti-chain rule".

Motivation: Let's say we want to calculate
$$\int 2x e^{x^2} dx.$$

If you spend enough time thinking, you might see that $F(x) = e^{x^2}$ is an antiderivative of $f(x) = 2xe^{x^2}$
Since $F'(x) = e^{x^2} \cdot 2x$ by the chain rule.
So $\int 2xe^{x^2} dx = e^{x^2} + C$.

Let's think of a systematic way to "undo" the chain rule.

The Method of Substitution:

Let f, g be diff'ble fcns, and let F be an antiderivative of F. Then $(F \circ g)'(x) = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x)$ and therefore
$$\boxed{\int f(g(x)) \cdot g'(x)\, dx = F(g(x)) + C.}$$

Now let $u = g(x)$. Then $\frac{du}{dx} = g'(x)$. So the eqn in the red box becomes
$$\int f(u) \cdot \left(\frac{du}{dx} dx = F(u) + C\right.$$
$$\Rightarrow \boxed{\int f(u)\, du = F(\; + C}$$

$u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2x\,dx$

Going to the previous example:
$$\int \underbrace{e^{x^2}}_{f(u)} \cdot \underbrace{2x\,dx}_{du} = \int e^u du = e^u + C = e^{x^2} + C$$
$u = x^2$

Remark: Since we are changing the "dummy variable" from x to u, this is sometimes called a "change of variables".

Q? How can you recognize when to use the method of substitution?

Recall: Last quarter, we talked about how you have to use the chain rule when you have a composition of fcns. I.e., when it looks like one fcn (the "inside" fcn) plugged into another fcn (the "outer" fcn). Since substitution is the "anti-chain rule", we can try to do it when we have a composition. However, it only works if the derivative of the "inside" fcn shows up.

Examples: ① $\int (2x+1)(x^2+x)^4\,dx$

Here, we see the composition $h(x) = (x^2+x)^4$ where the "inside" fcn is $u(x) = x^2+x$ and the "outside" fcn is $g(x) = x^4$.

($(g \circ u)(x) = g(u(x)) = (x^2+x)^4$.)

Since the derivative of the inside fcn is $u'(x) = 2x+1$, which shows up in the integral, we can substitute: Let $u = x^2+x$ be the "inside" fcn. Then

$$\frac{du}{dx} = 2x+1 \quad \text{and so}$$

$$\int (2x+1)(x^2+x)^4\,dx = \int u^4 \left(\frac{du}{dx}\right) dx = \int u^4\,du = \frac{u^5}{5} + C$$

$$\boxed{= \frac{(x^2+x)^5}{5} + C}$$

② $\int \sqrt{\sin(3t)}\cos(3t)\,dt$.

Here we have the composition $h = \sqrt{\sin(3t)}$ where the "inner" fcn is $u = \sin(3t)$ and the "outer" fcn is $f = \sqrt{t}$. The derivative of the "inner" fcn is

$$\frac{du}{dt} = \cos(3t)\cdot 3 \Rightarrow \frac{1}{3}du = \cos(3t)\,dt$$

which pretty much shows up in the integrand. (We are actually off by a factor of 3, but constants don't matter.). So

$$\int \sqrt{\sin(3t)}\cos(3t)\,dt = \int \sqrt{u}\left(\frac{1}{3}du\right) = \frac{1}{3}\int \sqrt{u}\,du$$
$$= \frac{1}{3}\int u^{1/2}\,du$$
$$= \frac{1}{3}\left(\frac{u^{3/2}}{\left(\frac{3}{2}\right)}\right) + C$$
$$= \frac{2}{9}u^{3/2} + C$$
$$= \frac{2}{9}\left(\sin(3t)\right)^{3/2} + C$$

③ $\int \frac{\ln(t)}{t}\,dt$

This time we don't really see a composition of fcns. Nevertheless, we see $u = \ln(t)$, and its derivative:

$$\frac{du}{dt} = \frac{1}{t} \Rightarrow du = \frac{1}{t}\,dt$$

$$\int \frac{\ln(t)}{t}\,dt = \int u\,du = \frac{u^2}{2} + C = \frac{(\ln(t))^2}{2} + C$$

④ $\int \tan(\theta)\,d\theta$.

Well, there is no composition, and the derivative of $f(\theta) = \tan(\theta)$ is $\frac{1}{\cos^2(\theta)}$, which doesn't show up. So let's do some work first. Recall, $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$.

$$\Rightarrow \int \tan(\theta)\,d\theta = \int \frac{\sin(\theta)}{\cos(\theta)}\,d\theta.$$

Now we have a composition $h = \frac{1}{\cos(\theta)}$ where the "inside" fcn is $u = \cos(\theta)$ and the outside fcn is $f(\theta) = \frac{1}{\theta}$.

Then $\frac{du}{d\theta} = -\sin(\theta) \Rightarrow -du = \sin(\theta)d\theta$

So
$$\int \frac{\sin(\theta)}{\cos(\theta)}\,d\theta = \int \frac{1}{\cos(\theta)}\left(\sin(\theta)\right)d\theta = \int \frac{1}{u}(-du)$$
$$= -\int \frac{1}{u}\,du$$
$$= -\ln|u| + C$$
$$= -\ln|\cos(\theta)| + C$$

Substitution and the bounds of integration:

When computing definite integrals using substitution, you have two options:

① Plug in the bounds at the very end, after you substitute the original variable back in, or

② Change the bounds when you substitute.

Example: ① $\int_1^4 \frac{e^{\sqrt{y}}}{\sqrt{y}}\,dy.$

We see the composition $h = e^{\sqrt{y}}$ where $u = \sqrt{y}$ is the "inner" fcn. Then
$$\frac{du}{dy} = \frac{d}{dy}\left(y^{1/2}\right)$$
$$= \frac{1}{2}y^{-\frac{1}{2}}$$
$$= \frac{1}{2\sqrt{y}}$$

$$\Rightarrow 2\,du = \frac{1}{\sqrt{y}}\,dy$$

which shows up in the integrand, so we can substitute.

Let's first change the bounds as we substitute.

The original bounds are $y=1$ to $y=4$. When $y=1$, $u=\sqrt{1}=1$, and when $y=4$, $u=\sqrt{4}=2$. So the new bounds are $u=1$ to $u=2$.

$$\int_{y=1}^{y=4} \frac{e^{\sqrt{y}}}{\sqrt{y}}\,dy = \int_{u=1}^{u=2} e^u(2\,du) = 2\int_{u=1}^{u=2} e^u\,du$$

$$= 2e^u\Big|_{u=1}^{u=2}$$

$$= \boxed{2e^2 - 2e^1}$$

Alternatively, we could plug in the bounds at the end:

$$\int_{y=1}^{y=4} \frac{e^{\sqrt{y}}}{\sqrt{y}}\,dy = \int_{u(1)}^{u(4)} 2e^u\,du = 2e^u\Big|_{u(1)}^{u(4)} = 2e^{\sqrt{y}}\Big|_{y=1}^{y=4}$$

$$= 2e^{\sqrt{4}} - 2e^{\sqrt{1}}$$

$$= \boxed{2e^2 - 2e^1}$$

As you see, we get the same answer.

Remark: In the 2^{nd} method, I wrote the bounds as $\int_{u(1)}^{u(4)}$ when I substituted. This notation is a placeholder. $u(1)$ and $u(4)$ mean "whatever u is when $y=1$ and $y=4$".

(2) $$\int_0^{\frac{\pi}{4}} \frac{\tan^2(\theta)}{\cos^2(\theta)}\,d\theta$$

Here, if $u=\tan(\theta)$, then $\frac{du}{d\theta} = \frac{1}{\cos^2(\theta)} \Rightarrow du = \frac{1}{\cos^2(\theta)}\,d\theta$

which shows up in the integrand.

$$\text{So} \quad \int_{\theta=0}^{\theta=\frac{\pi}{4}} \frac{\tan^2(\theta)}{\cos^2(\theta)}\, d\theta = \int_{\theta=0}^{\theta=\frac{\pi}{4}} (\tan(\theta))^2 \left(\frac{1}{\cos^2(\theta)}\, d\theta\right)$$

$$= \int_{u(0)}^{u(\frac{\pi}{4})} u^2\, du$$

$$= \frac{u^3}{3}\Bigg|_{u(0)}^{u(\frac{\pi}{4})}$$

$$= \frac{(\tan(\theta))^3}{3}\Bigg|_{\theta=0}^{\theta=\frac{\pi}{4}}$$

$$= \frac{(\tan(\frac{\pi}{4}))^2}{3} - \frac{(\tan(0))^3}{3}$$

$$= \frac{1}{3} - \frac{0}{3}$$

$$\boxed{= \frac{1}{3}}$$