Longo: Math 10B - Winter 2017
Lecture Notes

Date: January 9, 2017

Section:
- [ ] §5.1

Topics Covered:
- [ ] History / Motivation
- [ ] How do measure determine distance travelled from speed

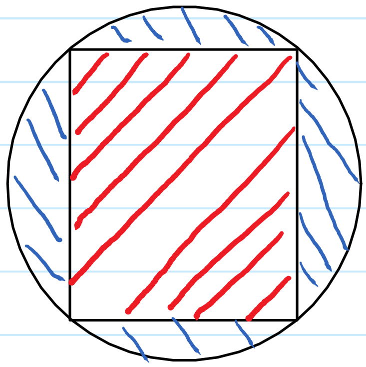
§5.1: How to measure distance travelled from speed.

Historical Prelude:

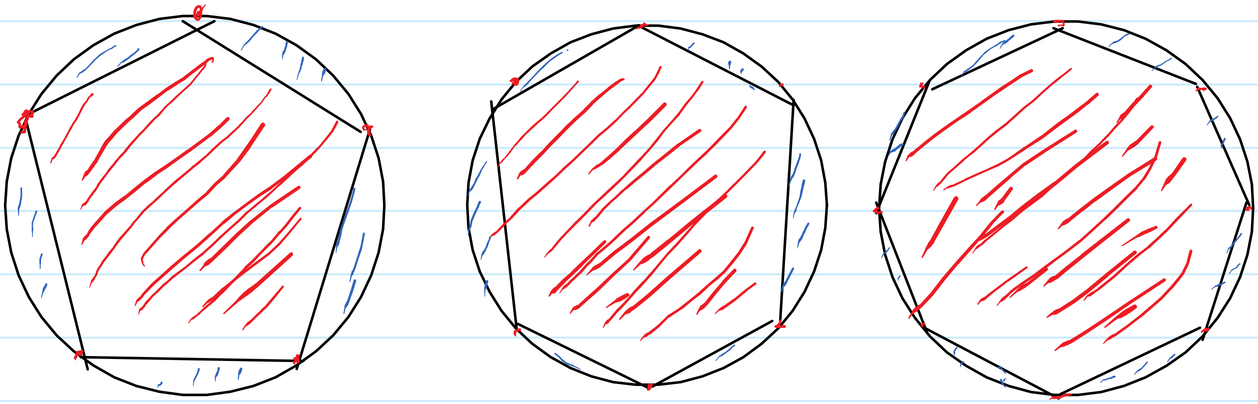
Historically integration (the focus of this course) originated long before differentiation.

Archimedes' Method of Exhaustion (~300 BC).

Let's pretend we want to find the area of a circle (and that we don't already know the formula).



Let A_{circ} be the area of the circle. As a first step, we can inscribe a square into the circle. The area of the square is close to A_{circ}, but we get a better approx. by inscribing regular polygons with more and more sides.

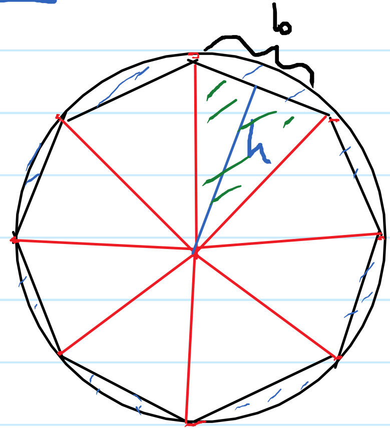


Let n be the number of sides of the polygon, and A_n be the area of polygon. (Red area)

Archimedes noticed that if n gets bigger and bigger, A_n gets closer and closer to $A_{circ.}$ He called this the Method of Exhaustion. In modern language, we say

$$\boxed{\lim_{n\to\infty} A_n = A_{circ.}}$$

Q? What is A_n?



To calculate A_n, split the polygon into n-many equal triangles with base b, and height, h.

If n is big, $h \approx r$ (radius of circle).
and $b \approx \dfrac{(\text{circumference})}{n} = \dfrac{2\pi r}{n}.$

$$\text{Since} \quad A_n = \underbrace{\left(\tfrac{1}{2} b h\right)}_{\text{Area of triangle}} \cdot \underbrace{n}_{\#\text{ of triangles.}}$$

z

$$A_n \approx \left(\left(\tfrac{1}{2}\right)\left(\frac{2\pi r}{n}\right)(r)\right) \cdot n$$

$$= \pi r^2.$$

So $\boxed{A_{circ} = \lim_{n\to\infty} A_n = \pi r^2}$ as expected.

Summary: Split the shape into n-many pieces so that it is easy to find the area of each piece. Then if we let $n \to \infty$, we find the area we want.

We now call this process integration.
2,000 years later, the founders of calculus found an intimate relationship between integration and differentiation. To see the connection, let's consider velocity.

§5.1: Measuring distance from speed.

Suppose a car is driving across the country. Let $s(t)$ be the function that tells us the total distance travelled (in miles) after t hours.

As we saw in Math 10a, the derivative of s, $v(t) = s'(t) = \frac{ds}{dt}$ tells us the velocity of the car (in mph) at time t.

Idea: If we know the speed of the car, we should be able to calculate the total distance traveled.

Simple Example: Suppose we know the car travels at a constant speed of 70 mph.
I.e., $s'(t) = 70$.

Then after 4 hours, the car will have

travelled 280 miles.

Harder example: Now suppose we have the table of values:

t	0	0.5	1	1.5	2
$s'(t)$	0	40	60	65	70

Notice: $v(t)$ is increasing and $v(t) \geq 0$ for all t.

we don't know the exact speed every second, but we know, for example, that between 0.5 and 1 hours, the car is going between 40 and 60 mph; between 1 and 1.5 hours, the car is going between 60 and 65 mph, and so on.

As a lower bound, we can say

$$\text{distance travelled} \geq \underbrace{\left(\tfrac{1}{2}\right)}_{\text{time elapsed}}\underbrace{(0)}_{\text{lower bound for speed}} + \left(\tfrac{1}{2}\right)(40) + \left(\tfrac{1}{2}\right)(60) + \tfrac{1}{2}(65)$$
$$= 2 + 30 + 32.5$$
$$= 82.5$$

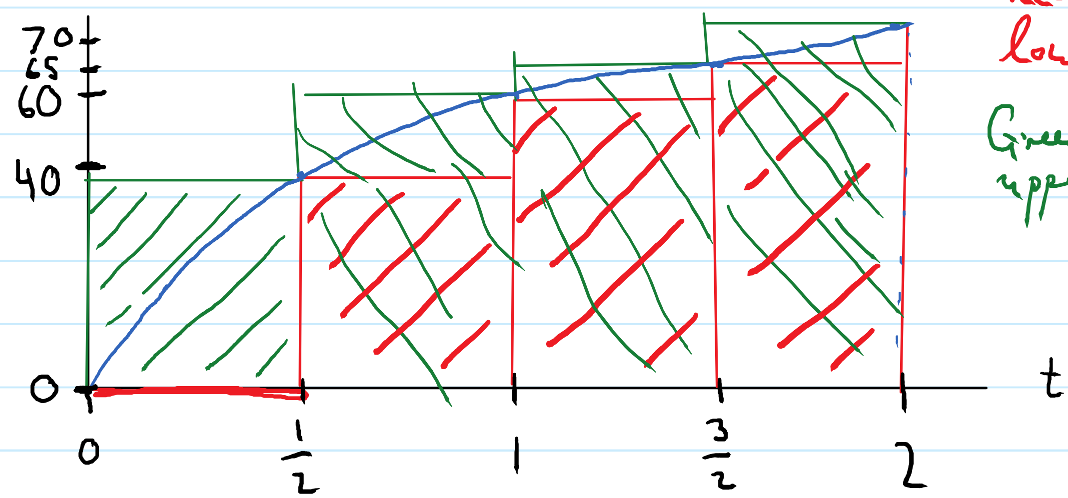
As an upper bound, we have

$$\text{distance travelled} \leq \underbrace{\left(\tfrac{1}{2}\right)}_{\text{time elapsed}}\underbrace{(40)}_{\text{upper bound for speed}} + \left(\tfrac{1}{2}\right)(60) + \left(\tfrac{1}{2}\right)(65) + \left(\tfrac{1}{2}\right)(70)$$

$$= 20 + 30 + 32.5 + 35$$
$$= 117.5$$

So the car travelled between 82.5 and 117.5 miles.

$\boxed{Q?}$ What does this mean graphically?



Red area = lower bound.

Green area = upper bound.

Notice: ① The area of each rectangle is

base × height

= (time elapsed) × (Velocity at a certain time)

$= (\Delta t) \times (v(t_*))$ or $(\Delta t) \times (v(t^*))$

Notation: t_* means the t-value that gives a lower bound

t^* means the t-value that gives an upper bound

② Both the red area and the green area are roughly the area under the velocity curve (blue).

Idea:

If we include more data points, the estimation will be more accurate.

Suppose we have:

t	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
$v(t)$	0	25	40	55	60	62	65	67	70

lower bound:

$$\text{distance travelled} \geq \left(\tfrac{1}{4}\right)(0)+\left(\tfrac{1}{4}\right)(25)+\tfrac{1}{4}(40)+\tfrac{1}{4}(55)+\left(\tfrac{1}{4}\right)(60)+\left(\tfrac{1}{4}\right)(62)+\left(\tfrac{1}{4}\right)(65)+\left(\tfrac{1}{4}\right)(67)$$

$$=\left(\tfrac{1}{4}\right)\left(25+40+55+60+62+65+67\right)$$

$$= 93.5$$

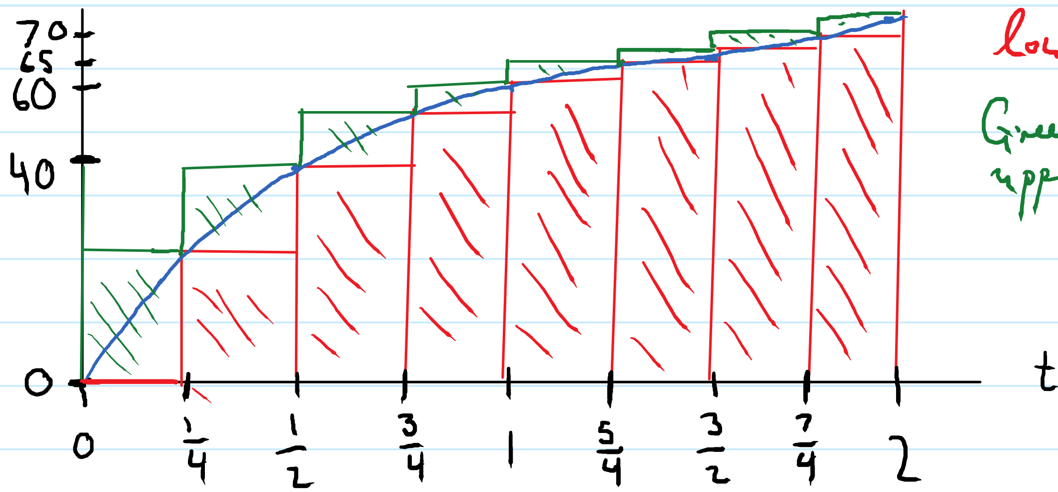
Upper bound:

$$\text{distance travelled} \leq \left(\tfrac{1}{4}\right)(25)+\tfrac{1}{4}(40)+\tfrac{1}{4}(55)+\left(\tfrac{1}{4}\right)(60)+\left(\tfrac{1}{4}\right)(62)+\left(\tfrac{1}{4}\right)(65)+\left(\tfrac{1}{4}\right)(70)$$

$$= 111$$

(More Accurate!)

Graphically:



Red area = lower bound.

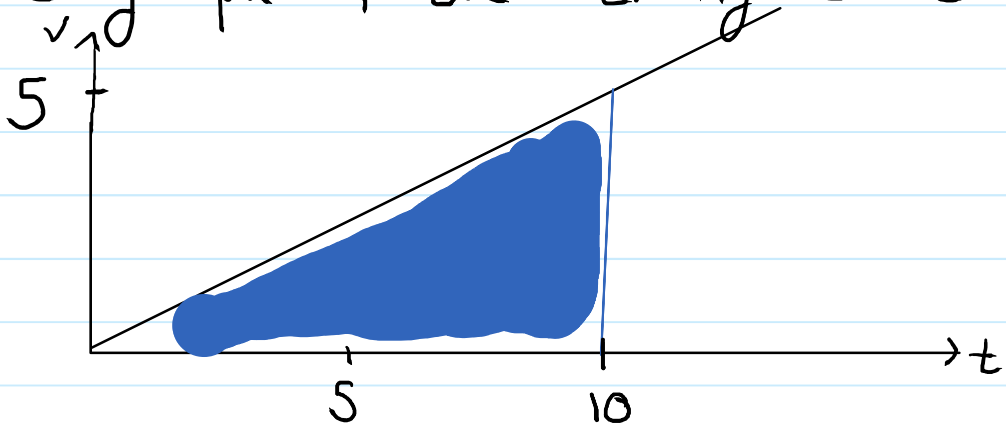
Green area = upper bound.

As we can see, when we partion the interval $[0,2]$ into more and more pieces, the estimates are more and more accurate. Furthermore, the red and blue areas, which represent the lower and upper bounds for distance travelled respectively, are getting closer and closer to the area under the curve.

Conclusion: If velocity is positive (meaning the object is always moving forward), then the total distance travelled is the area under the velocity curve.

Example: Suppose the velocity of a car after t seconds is given by $v(t) = \frac{t}{2}$ ft/s. How far does the car travel in the first 10 seconds?

Sol: The graph of the velocity curve is:



The area between the t-axis, and the velocity curve from 0 to 10 seconds is a triangle with base 10 and height 5. The distance travelled is the area of the triangle:

$$\begin{aligned} \text{Area} &= \tfrac{1}{2}bh \\ &= \left(\tfrac{1}{2}\right)(5)(10) \\ &= 25. \end{aligned}$$

The car travelled 25 feet.

Negative velocity and displacement:

Q? What happens if velocity is negative? (i.e. the car travels backwards).

Example: Suppose a car drives 60 mph east for 2 hours. Then the car turns around and drives west at 80 mph for

1 hour. What is the car's total displacement after 3 hours?

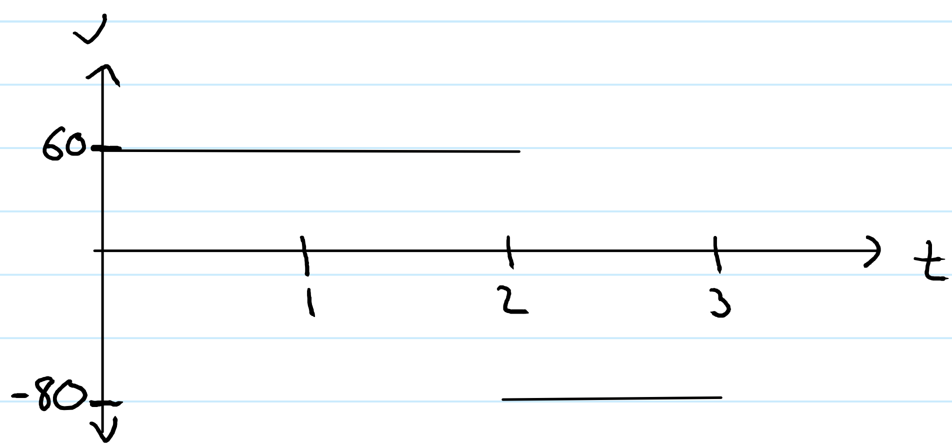
Def: Total displacement means the distance from the objects initial position to its final position.

Sol: The car's velocity is given by the piecewise fcn:

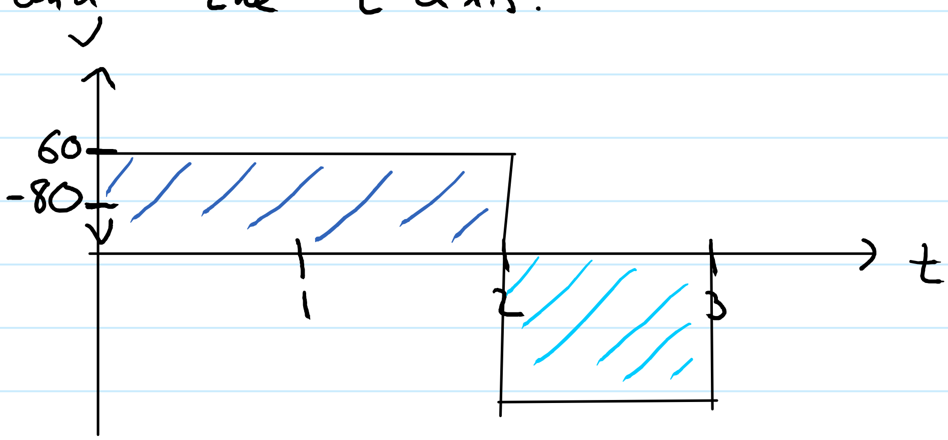
$$v(t) = \begin{cases} 60 & \text{if } 0 \le t < 2 \\ -80 & \text{if } t \geq 2 \end{cases}$$

Note: positive velocity means eastward motion, and negative velocity means westward motion.

The graph of velocity from 0 to 3 hours is:



We compute areas between the velocity curve and the t-axis:



The dark blue shape is a rectangle with height 60 and base 2, so the dark blue area is 120.
The light blue shape is a rectangle with height 80 and base 1, so the light blue area is 80

Since the light blue area is below the t-axis, we will count this as "negative area". This means the car moved backwards (west) by 80 miles.

The total displacement is the "signed" area under the curve:

(distance travelled forward)
− (distance travelled backwards)

= (dark blue area)
− (light blue area)

= 120 − 80
= 40

Since this number is positive, the total displacement is 40 miles EAST.

Conclusion: The total displacement of a moving object is the "signed" area under the velocity curve.