Longo: Math 10B - Winter 2017
Lecture Notes

Date: January 11, 2017

Section:
- [ ] §5.2

Topics Covered:
- [ ] Riemann sums
- [ ] The definition of the definite integral

§5.2: The definite integral:

Sigma notation: We will be computing sums all quarter, so we need a good notation. We use the capitl Greek letter Sigma, Σ.

Note: Sigma is the Greek letter 'S' for "sum"

Suppose I have a sequence of N-many numbers

$$m_1, m_2, \ldots, m_N$$

Then instead of writing $m_1 + m_2 + \ldots + m_N$, we write

$$\boxed{\sum_{i=1}^{N} m_i := m_1 + m_2 + m_3 + \ldots + m_N}$$

Here, i is called the index and for each $i = 1, 2, 3, \ldots, N$, m_i is called the i^{th} summand, or the i^{th} term. With this notation, the "$i=1$" tells us we start at $i=1$ and end at $i=N$. If we wanted to start at the 2^{nd} term, i.e., if we want $m_2 + m_3 + \ldots + m_N$, we could write $\sum_{i=2}^{N} m_i$.

Note: There is nothing special about the letter i. We could also write $\sum_{j=1}^{N} m_j$ or $\sum_{k=1}^{N} m_k$.

We commonly write some expression or function involving the index i for the summand.

Examples: (1) $\sum_{i=1}^{5} i = 1+2+3+4+5$.

② $\sum_{i=3}^{7} \frac{i^2}{2} = \frac{3^2}{2} + \frac{4^2}{2} + \frac{5^2}{2} + \frac{6^2}{2} + \frac{7^2}{2}$

③ $\sum_{n=0}^{4} \frac{2^n}{n!} = \frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!}$

$= 1 + \frac{2}{1} + \frac{4}{2} + \frac{8}{6} + \frac{16}{24}$

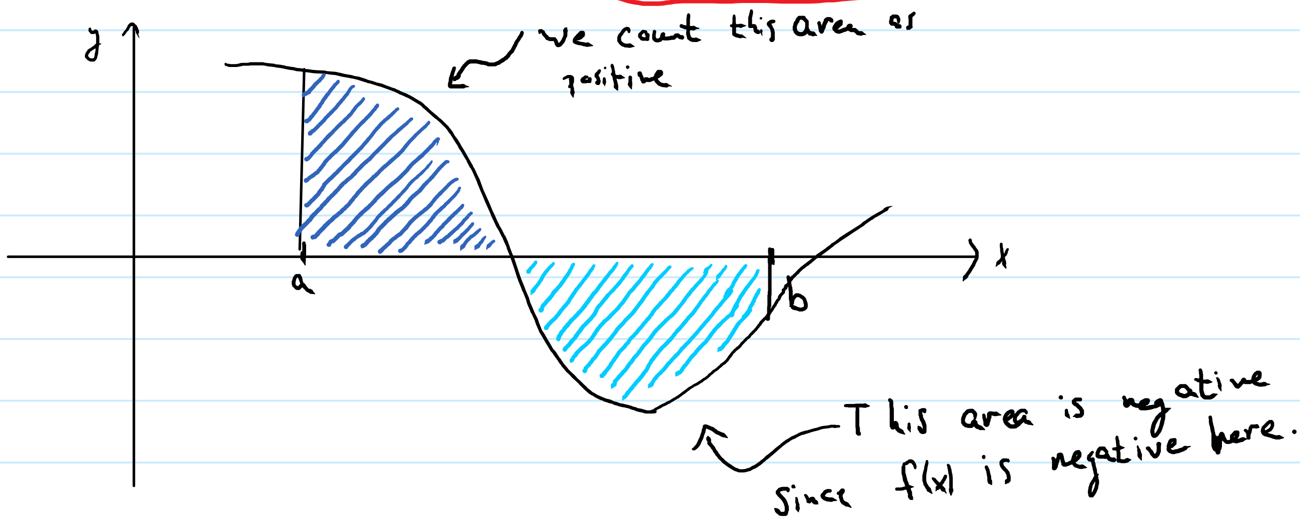
(Recall: $n! = n\cdot(n-1)\cdot(n-2)\cdot \ldots \cdot 3\cdot 2\cdot 1$, and $0! = 1$)

④ If $f(x) = \sqrt{x}$, then $\sum_{i=1}^{3} f(i) = f(1) + f(2) + f(3)$
$= \sqrt{1} + \sqrt{2} + \sqrt{3}$

Note: We will see later that $\sum_{n=0}^{4} \frac{2^n}{n!}$ is very close to e^2

Let f be a continuous function on the interval $[a,b]$. Using the last lecture as motivation, we wish to compute the "signed area" between the graph of f and the x-axis.

Recall: "signed area" means we consider area above the x-axis as positive and area below the x-axis is "negative area".



Definition: The signed area between the graph of $f(x)$ and the x-axis is called the definite integral of f from a to b and is denoted

$$\int_a^b f(x)\,dx$$

(we discuss this notation shortly)

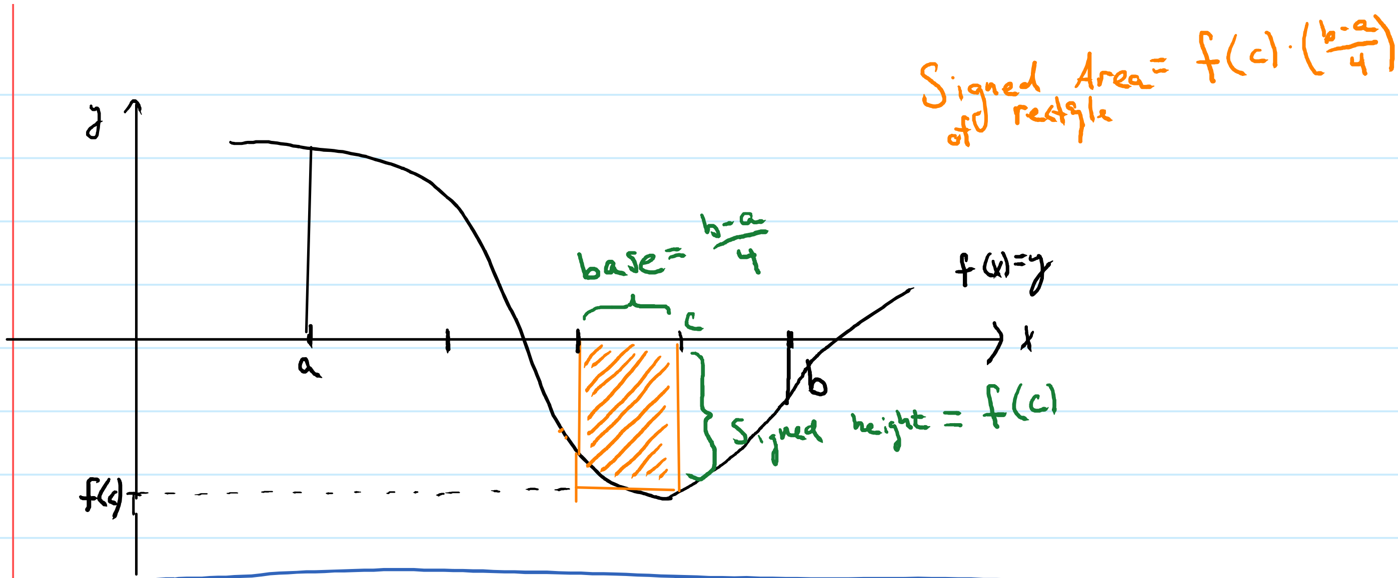
As before, we partition the interval $[a,b]$ into equal pieces and approximate using rectangles:

The definite integral of f from a to b is roughly (red area) − (orange area).

As before, if we increase the number of rectangles, the approximation will be more accurate.

Q? How do we compute the (signed) area of the rectangles?

Riemann Sums: In the picture above, the (signed) height of each rectangle is the value of f at the right hand side of base of the rectangle, and the base of the rectangle is the length of the interval, b−a, divided by the number of rectangles



Notation: ① For $i=1,2,\ldots,N$, where $N=$ (# of rectangles) let x_{r_i} be the x-value on the right hand side of the i^{th} triangle.

② Let $\Delta x = \frac{b-a}{N}$.

Remark: Δx is meant to indicate "change in x". Think about the velocity example from last time!

The (signed) area of the rectangles is the sum of the (signed areas of the rectangles. This area is called the Right-handed Riemann Sum:

$$f(x_{r_1})\Delta x + f(x_{r_2})\Delta x + \ldots + f(x_{r_N})\Delta x$$

$$=$$

$$\sum_{i=1}^{N} f(x_{r_i})\Delta x$$

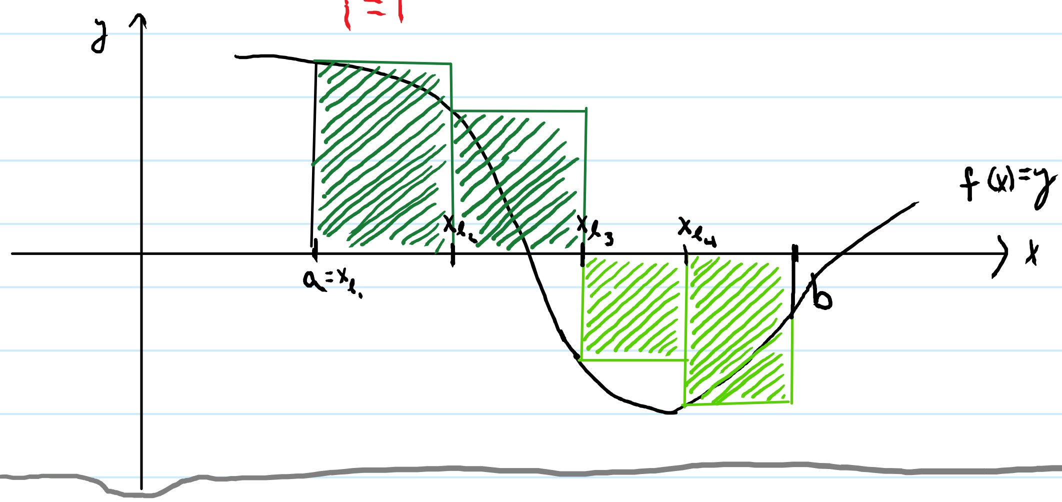
Notice we also could have used the x-value on the left hand side of each subinterval.

If we let x_{l_i} be the x-value on the left side of the i^{th} subinterval, we get the Left-hand Riemann Sum:

$$f(x_{\ell_1})\Delta x + f(x_{\ell_2})\Delta x + \dots + f(x_{\ell_N})\Delta x$$

$$=$$

$$\sum_{i=1}^{N} f(x_{\ell_i})\Delta x.$$



As N (the number of rectangles) $\longrightarrow \infty$, the left and right hand Riemann sums both get closer and closer to the (signed) area under the curve. This gives us the

Definition of the definite integral: Let f be a continuous fcn on the interval $[a,b]$. The definite integral of f from a to b is defined by

$$\int_a^b f(x)\,dx = \lim_{N\to\infty}\left(\sum_{i=1}^{N} f(x_{r_i})\,\Delta x\right) \qquad \text{Right-hand Riemann Sum}$$

$$= \lim_{N\to\infty}\left(\sum_{i=1}^{N} f(x_{\ell_i})\,\Delta x\right) \qquad \text{Left-hand Riemann Sum.}$$

Remarks: (1) The $\int$ is a big 'S', for "sum"

(2) f is called the integrand

(3) a and b are called the limits of integration

④ The 'x' is called the dummy variable and has no real significance. We could change it to whatever we want: $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

$$= \int_a^b f(☺)\,d☺$$

⑤ The 'dx' comes from the Δx.

! Warning. Do Not forget the dx.

" $\int_a^b f(x)$ " is meaningless!

Example: ① Compute the left and right hand Riemann sums with $N=2$ for $\int_0^2 \sqrt{4-x^2}\,dx$

② Find the exact value of the integral.

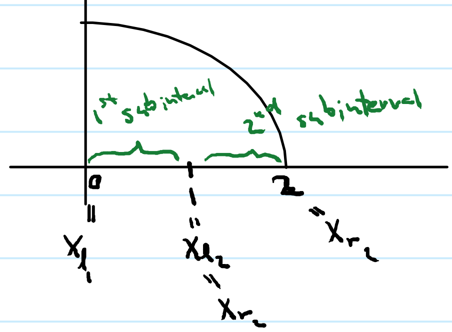
Sol: Before we start, lets graph $f(x)=\sqrt{4-x^2}$ from 0 to 2.

Notice that if $\quad y=\sqrt{4-x^2}$

$$\Rightarrow \quad y^2 = 4-x^2$$

$$\Rightarrow \quad x^2+y^2=4.$$

This is a circle centered at the origin with radius 2. Since $f(x)\geq 0$, we are only looking at the upper half.



① Notice $x_{l_1}=0, \quad x_{r_1}=1$

$x_{l_2}=1, \quad x_{r_2}=2.$

So we have:

$$\text{Left-hand sum} = \sum_{i=1}^{2} f(x_{\ell_i})\Delta x$$
$$= f(x_{\ell_1})\cdot 1 + f(x_{\ell_2})\cdot 1$$
$$= f(0)\cdot 1 + f(1)\cdot 1$$
$$= (\sqrt{4})\cdot 1 + (\sqrt{4-1^2})\cdot 1$$
$$= \boxed{2+\sqrt{3}}$$

$$\text{Right-hand sum} = \sum_{i=1}^{2} f(x_{r_i})\Delta x$$
$$= f(x_{r_1})\cdot 1 + f(x_{r_2})\cdot 1$$
$$= f(1)\cdot 1 + f(2)\cdot 1$$
$$= (\sqrt{4-1})\cdot 1 + (\sqrt{4-2^2})\cdot 1$$
$$= \boxed{\sqrt{3}}$$

2 To calculate $\int_0^2 \sqrt{4-x^2}\,dx$, we calculate the area under the curve. Since this is a quarter circle with radius 2, we have

$$\boxed{\int_0^2 \sqrt{4-x^2}\,dx = \frac{1}{4}(2^2\pi) = \frac{1}{4}\cdot(4\pi) = \pi}$$

Remark: $2+\sqrt{3} \approx 3.7$, which is fairly close to π. So the left-hand sum was a decent approximation.