RESEARCH STATEMENT

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1. Introduction

Let $\mathcal{G}$ be a finite graph and let $V(\mathcal{G})$ be its vertex set. For a subset X of the vertices of $\mathcal{G}$ define

$$\partial(X) := \{y \in V(\mathcal{G}) | \exists x \in X \text{ such that } x \text{ is connected to } y\}$$

to be the *boundary of* X. Define the *Cheeger constant*, $h(\mathcal{G})$, of $\mathcal{G}$ by

$$h(\mathcal{G}) = \min_{X \subset V(\mathcal{G}), 2|X| \leq |V(\mathcal{G})|} \frac{|\partial(X)|}{|X|}.$$

If ε is a positive constant, a family of graphs $\{\mathcal{G}_i\}_{i\in I}$ is called a *family of* ε*-expanders* if $\inf_{i\in I} h(\mathcal{G}_i) > \varepsilon$. That is, if the Cheeger constants $h(\mathcal{G}_i)$ are uniformly bounded away from zero by ε.

Historically, explicit examples of families of expander graphs have been difficult to construct. The earliest constructions arose as Cayley-Schreier graphs of lattices of Lie groups of higher rank. Let us state the following definitions:

Definition 1 (Cayley-Schreier Graphs)**.** *Let G be a finite group, H a subgroup of G, and Ω a symmetric subset of G. The **Cayley-Schreier graph** of G with respect to H and Ω,* $\mathrm{Sch}(G, H, \Omega)$, *is defined to be the graph whose vertices coincides with the coset space G/H where the vertices gH and $g'H$ are connected exactly when there exists an element $\omega \in \Omega$ with $gH = \omega g'H$. The **Cayley** graph of G with respect to Ω,* $\mathrm{Cay}(G, \Omega)$, *is defined to be* $\mathrm{Sch}(G, \langle id\rangle, \Omega)$.

Definition 2 (Kazhdan's Property (T))**.** *Let Γ be a discrete group and $\Omega \subset \Gamma$ be a set of generators. Γ is said to have **Kazhdan Property (T)** if there exists a constant $\varepsilon > 0$ with the following property. For any unitary representation $\rho : \Gamma \to U(H)$ with no nonzero fixed vectors from Γ into the group of unitary operators of a Hilbert space H, and for any nonzero vector $v \in H$, there exists $\omega \in \Omega$ such that*

$$\|\rho(\omega)v - v\| \geq \varepsilon\|v\|.$$

In 1973 ([16]), Margulis proved that if $\Gamma = \langle\Omega\rangle$ is a group with Kazhdan property (T), then the family of Cayley graphs

$$\{\mathrm{Cay}(\Gamma/N_i, \Omega N_i/N_i)\}_{N_i \triangleleft \Gamma,\ [\Gamma:N_i]<\infty}$$

is a family of ε-expander graphs for some $\varepsilon > 0$. This, combined with Kazhdan's 1967 result which states that any lattice Γ in a simple Lie group of real rank at least 2 has property (T) ([11]), gave us a fairly rich source of examples.

The question then became: "What can we say about the rank 1 case?" Selberg's theorem implies ([22]) that if $\Omega \subset \mathrm{SL}_2(\mathbb{Z})$ is a finite symmetric subset which generates a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, then there exists a positive number ε such that the family of graphs

$$\{\mathrm{Cay}(\mathrm{SL}_2(\mathbb{F}_p)), \pi_p(\Omega))\}_{p \text{ prime}}$$

is a family of $\varepsilon > 0$-expander graphs where

$$\pi_p : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$$

denotes the reduction modulo p map. Lubotzky questioned whether or not the result is true when Ω generates a *thin* subgroup of $\mathrm{SL}_2(\mathbb{Z})$, i.e., a subgroup of infinite index which is dense in the Zariski topology. In particular, his famous "$1-2-3$ problem" asks if families of graphs

$$\{\mathrm{Cay}(\mathrm{SL}_2(\mathbb{F}_p), \pi_p(\Omega_i))\}_{p>3 \text{ prime}}$$

form a families of expanders where

$$\Omega_i := \left\{ \begin{pmatrix} 1 & \pm i \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm i & 1 \end{pmatrix} \right\}, \ i = 1, 2, 3.$$

Here Ω_1 generates $\mathrm{SL}_2(\mathbb{Z})$ while Ω_2 generates a finite index subgroup. Ω_3, however, generates a thin subgroup of $\mathrm{SL}_2(\mathbb{Z})$. This question was settled by Bourgain and Gamburd in 2008 [3] where they showed that for a subset Ω of $\mathrm{SL}_2(\mathbb{Z})$ the family of graphs

$$\{\mathrm{Cay}(\mathrm{SL}_2(\mathbb{F}_p), \pi_p(\Omega)\}_{p \text{ prime}}$$

is a family of expanders if and only if $\langle\Omega\rangle$ is not virtually solvable. Their method of proof, the so called "Bourgain-Gamburd Machine," has since been used in [2], [4], [5], [6], [23], and finally in [20] where the authors found necessary and sufficient conditions for such a contruction to yield a family of expander graphs. Namely, they proved the following:

Theorem 3. *Let $\Gamma \subset \mathrm{GL}_d(\mathbb{Z}[1/q_0])$ be the group generated by a symmetric set Ω. Then*

$$\{\mathrm{Cay}(\Gamma/\Gamma(q), \Omega\Gamma(q)/\Gamma(q)\},$$

where $\Gamma(q) := Ker(\Gamma \to \Gamma(mod\ q))$ is the kernel of the reduction modulo q map, is a family of expander graphs as q ranges over the square free integers coprime to q_0 if and only if the connected component of the Zariski-closure of Γ is perfect.

This type of result has come to be known as "superstrong approximation" as it strengthens the idea of strong approximation in the sense of [18, Ch. 7]. Its applications in mathematics have proven to be deep and diverse; including: Appolonian circle packing, homogeneous dynamics, Zaremba's conjecture, and affine sieving (see [8] for an overview).

To date, not much has been shown for the analogous "superstrong approximation" question in positive characteristic.

2. Results

We consider the following question which is a positive characteristic variation of the main result in [20]. Let p be a prime number, $\mathbb{F}_p[t]$ be the ring of polynomials with coefficients in $\mathbb{F}_p$, and $\mathbb{F}_p(t)$ its field of fractions. Let Ω be a finite symmetric subset of $\mathrm{GL}_n(\mathbb{F}_p(t))$. Since $|\Omega| < \infty$, there exists a common denominator Q_0 of the entries of the matrices in Ω and we see that $\Omega \subset \mathrm{GL}_n(\mathbb{F}_p[t, 1/Q_0])$. Let $\Gamma = \langle\Omega\rangle$ be the group generated by Ω, and $\mathbb{G} := \overline{\Gamma}^{Zar.}$ be the Zariski-closure of Γ. For a polynomial $Q \in \mathbb{F}_p[t]$ coprime to Q_0, let

$$\pi_Q : \mathbb{F}_p[t, 1/Q_0] \to \mathbb{F}_p[t, 1/Q_0]/(Q) = \mathbb{F}_p[t]/(Q)$$

be the "reduction modulo Q" map. This gives rise to a homomorphism

$$\pi_Q : \Gamma \to \mathrm{GL}_n(\mathbb{F}_p[t]/(Q)).$$

Under these assumptions, what conditions are necessary and sufficient for the family of Cayley graphs

$$\{\mathrm{Cay}(\pi_Q(\Gamma), \pi_Q(\Omega))\}_{Q\in\mathbb{F}_p[t],\ Q \text{ square free},\ (Q,Q_0)=1}$$

to be a family of $\varepsilon-$expander graphs for some $\varepsilon > 0$?

The hope is that we have expanders if and only if the connected component of $\mathbb{G}$ is perfect, as is the case in characteristic zero. However, there are certain difficulties here that do not appear in characteristic zero, and so the previous techniques fall short. Despite these difficulties, we do have a result which holds under some additional assumptions. We first introduce some notation.

Let $\mathcal{P} \subset \mathbb{F}_p[t]$ be the set of all irreducible polynomials in $\mathbb{F}_p[t]$ and Σ be the set of all square free polynomials in $\mathbb{F}_p[t]$ with the property that no two irreducible factors of any polynomial $Q \in \Sigma$ have the same degree. For any constant c, let $\mathcal{P}_c$ (resp. Σ_c) be the set of polynomials in $\mathcal{P}$ whose degrees have no divisors less than c (resp. square free polynomials in Σ whose irreducible factors lie in $\mathcal{P}_c$). We have the following:

Theorem 4 (L., Salehi Golsefidy (2015))**.** *Let p, Ω, Γ, $\mathbb{G}$ and Q_0 be as above. Assume $p \geq 5$ and $\mathbb{G}$ is absolutely almost simple and simply connected. Assume further that the ring generated by the set $\mathrm{Tr}(\mathrm{Ad}(\Gamma))$ is all of $\mathbb{F}_p[t, 1/Q_0]$. Assume $pr_i(\Gamma)$ is Zariski-dense in $\mathbb{G}_i$ for all i. Then there exist positive constants c and ε such that the family of graphs,*

$$\{Cay(\pi_Q(\Gamma), \pi_Q(\Omega)\}_{Q\in\Sigma_c,\ (Q,Q_0)=1},$$

forms a family of ε-expander graphs.

I'll now give a brief explanation of the proof and describe where the difficulties lie. Let

$$\mu(\gamma) := \begin{cases} \frac{1}{|\Omega|} & \text{if } \gamma \in \Omega \\ 0 & \text{if } \gamma \notin \Omega \end{cases}$$

be the uniform probability measure on Γ supported by Ω and for each polynomial $Q \in \mathbb{F}_p[t]$ let $\pi_Q[\mu]$ be the induced probability measure on $\pi_Q(\Gamma)$. For any probability measure ν on a group G, we denote by $\nu^{(\ell)}$ the ℓ fold convolution of ν with itself. In our proof, we follow the so called "Bourgain-Gamburd machine" which was first used in the proof of the main theorem in [3]. The machine has three main components. First, one must show that a random walk on $\mathrm{Cay}(\pi_Q(\Gamma), \pi_Q(\Omega))$ has an exponentially small chance of landing in any coset of a proper subgroup of $\pi_Q(\Gamma)$. Applying this fact to the trivial subgroup gives us a nice upper bound on the ℓ^2 norm on $\pi_Q[\mu]^{(l)}$ for $l \sim log|\pi_Q(\Gamma)|$. Next, one shows that we can convolve $\pi_Q[\mu]^{(l)}$ with itself a finite number of times independent of Q so that the resulting measure is very close to equidistribution in the ℓ^2 norm. Finally, one must use the technique of Sarnak and Xue which first appeared in ([21]) in which one calculates a trace formula and exploits the fact that the groups $\pi_Q(\Gamma)$ are c-quasirandom in the sense of Gowers [9] for some constant c which does not depend on Q to achieve a uniform upper bound for the second largest eigenvalue in the spectrum of the adjacency matrices of the Cayley graphs. This was shown in ([1]) to be an equivalent condition for the family of graphs to be an expanding family.

There are two key differences in our problem compared to the previous work in characteristic zero: the subgroup structure of the groups $\pi_Q(\Gamma)$ and the fact that representations of $\mathbb{G}$ are not necessarily completely reducible. By the Strong Approximation Theorem ([24, Thm. 0.6]), we have that if Q is square free and coprime to Q_0, and if $\deg(P)$ is sufficiently large for every prime factor P of Q, then

$$\pi_Q(\Gamma) = \mathbb{G}_Q(\mathbb{F}_p[t]/(Q))$$

and

$$\mathbb{G}_Q(\mathbb{F}_p[t]/(Q)) = \Pi_{P|Q}\mathbb{G}_P(\mathbb{F}_p[t]/(P))$$

where $\mathbb{G}_Q$ (resp. $\mathbb{G}_P$) are the group schemes obtained from $\mathbb{G}$ by reducing the polynomials which define $\mathbb{G}$ as a variety modulo Q (resp. P). Therefore in order to understand the subgroup structure of $\pi_Q(\Gamma)$, we must understand subgroups of $\mathbb{G}_P(\mathbb{F}_p[t]/(P)) = \mathbb{G}_P(\mathbb{F}_{p^{\deg P}})$.

In the characteristic zero case, one only needs to consider the subgroup structure of the $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ points of an algebraic $\mathbb{F}_p$-ssubgroup of GL_n. The structure of such subgroups is fully described by Madhav Nori in [17] where it is shown that every subgroup can be approximated by the $\mathbb{F}_p$ points of a proper algebraic subgroup. For larger fields, the correct classification is given by Larsen and Pink in [12]. As a corollary of their work we show that if $\mathbb{G}_0$ is an absolutely almost simple group of adjoint type defined over a finite field $\mathbb{F}_q$ and if $H \subset \mathbb{G}_0(\mathbb{F}_q)$ is a maximal proper subgroup then either there exists a proper algebraic subgroup $\mathbb{H}$ of $\mathbb{G}_0$ defined over the algebraic closure of $\mathbb{F}_q$ with $H \subset \mathbb{H}$, or there exists a subfield $\mathbb{F}_{q'}$ and a model $\mathbb{G}_1$ of $\mathbb{G}_0$ defined over $\mathbb{F}_{q'}$ (i.e., $\mathbb{G}_1 \otimes_{\mathbb{F}_{q'}} \mathbb{F}_q = \mathbb{G}_0$) with

$$[\mathbb{G}_1(\mathbb{F}_{q'}) : \mathbb{G}_1(\mathbb{F}_{q'})] \subset H \subset \mathbb{G}_1(\mathbb{F}_{q'})$$

where $\mathbb{G}_1$ is a model of $\mathbb{G}_0$ over a some subfield $\mathbb{F}_{q'}$ of $\mathbb{F}_q$. Subgroups of the former type are called *structural subgroups* while subgroups of the latter type are called *subfield type subgroups*. In an attempt to establish the first step of the "Bourgain Gamburd Machine" we show that if a subgroup $H \subset \mathbb{G}_Q(\mathbb{F}_p[t]/(Q))$ has the property that if the image of H in $\mathbb{G}_P(\mathbb{F}_{p^{\deg P}})$ for each divisor P of Q is a structural subgroup, then contained in a maximal algebraic subgroup of $\mathbb{G}$. By using an effective Nullstellensatz argument we show that the set

$$\mathcal{L}_\delta(H) := \{h \in \mathbb{G}(\mathbb{F}_p[t, 1/Q_0]) \mid \pi_P(h) \in H \text{ and } \|h\| < [G : H]^\delta$$

of "small lifts" of $\pi_P(H)$ is contained in a proper algebraic subgroup of $\mathbb{G}$ and we can bootstrap this argument to show that the set of "small lifts" of H is contained in a proper algebraic subgroup of $\mathbb{G}$. Then, we construct a finite set of irreducible representations of $\mathbb{G}$ with the property that any algebraic subgroup $\mathbb{H}$ of $\mathbb{G}$ fixes a line in at least one of these representations. It is clear that for any algebraic subgroup $\mathbb{H}$ of $\mathbb{G}$, the line spanned by $\wedge^{\dim \mathbb{H}}\mathfrak{h}$ in $\wedge^{\dim \mathbb{H}}\mathfrak{g}$ is stable under $\wedge^{\dim \mathbb{H}} \mathrm{Ad}\, \mathbb{H}$ but not all of $\wedge^{\dim \mathbb{H}} \mathrm{Ad}\, \mathbb{G}$. Unfortunately the representation $\wedge^{\dim \mathbb{H}} \mathrm{Ad}$ is not completely irreducible since $\mathbb{G}$ is defined over a field of positive characteristic. Nevertheless, using the classification of irreducible representations of reductive groups given in [10] we show that one of the irreducible subquotients of a composition series of $\wedge^{\dim \mathbb{H}}\mathfrak{g}$ has the desired property. We then use a "ping-pong" argument to show that the probability that a word of length $l \ \log |\pi_Q(\Gamma)|$ has an exponentially small chance of fixing a line in any of these representations and therefor the chance of landing in a subfield type subgroup after a random walk on $\mathrm{Cay}(\pi_P(\Gamma), \pi_P(\Omega))$ is exponentially small.

Since the trivial subgroup is clearly of structural type, we already get a nice bound on the ℓ^2-norm of $\pi_Q(\mu)$. By adapting the proof of Varjú in [23], I show that second step of the "Bourgain-Gamburd" machine holds so long as the degrees of our polynomials have no small divisors. The idea is that if we cannot get the ℓ^2-norm of $\pi_Q(\mu)$ to "flatten out" in finitely many steps, then the results of [7], [19], and [3] imply that the measure must concentrate on a coset of a large proper subgroup. The measure cannot concentrate on a coset of a proper structural subgroup since that contradicts what we have already shown. Therefore it must concentrate on a coset of a large subgroup whose image in $\pi_P(\Gamma)$ is structural for some divisor P of Q. However, the restrictions on the degrees of the divisors of Q guarantee that no such subgroup exists. Finally due to [13], we can use the trick of Sarnak and Xue to achieve a uniform bound on the second largest eigenvalues of the linear operators

$$T_{\pi_Q[\mu]} : L^2(\mathbb{G}_Q(\mathbb{F}_p[t]/(Q))) \to L^2(\mathbb{G}_Q(\mathbb{F}_p[t]/(Q))),$$

$$f \mapsto \pi_Q[\mu] * f,$$

which shows that the Cayley graphs indeed form a family of expander graphs.

3. Future Directions

My studies have been focused on the theory of "superstrong approximation" in algebraic groups (mainly in positive characteristic). I am interested and willing to study any topic related to algebraic groups. In this section I will focus on my ambitions in "superstrong approximation."

Expansion Without Condition. My immediate research goal is to try to strengthen the result of Theorem 4. I would like to prove the following

Question 1. *If Ω, Γ, and $\mathbb{G}$ are as in the hypothesis of Theorem 4, then is the family of graphs*

$$\{\mathrm{Cay}(\pi_Q(\Gamma), \pi_Q(\Omega))\}_{Q \text{ square free, } (Q,Q_0)=1}$$

a family of ε-expander graphs for some $\varepsilon > 0$?

I plan on attacking this problem in two steps. The core of the issue is that I currently do not know how to "escape" from proper subfield type subgroups after a random walk of $\deg P$ steps on the Cayley graph $\mathrm{Cay}(\pi_P(\Gamma), \pi_P(\Omega))$ for P irreducible and coprime to Q_0. A second issue that the process of bootstrapping the problem of "escaping" proper subgroups of $\pi_Q(\Gamma)$ for Q square free and coprime to Q_0 from the irreducible case is hindered by the existence of certain diagonal subgroups. For example if P_1 and P_2 are irreducible or the same degree, then there exists an isomorphism

$$\alpha : \mathbb{G}_{P_1}(\mathbb{F}_p[t]/(P_1)) \to \mathbb{G}_{P_2}(\mathbb{F}_p[t]/(P_2))$$

and a subgroup

$$H := \{(g, \alpha(g)) : g \in \mathbb{G}_{P_1}(\mathbb{F}_p[t]/(P_1))\} \subset \mathbb{G}(\mathbb{F}_p[t]/(Q)).$$

Since the projection to each factor is surjective, I can't use the same argument as before to show that these subgroups can be avoided. I do believe that if I can show that we can "escape" from subfield type subgroups, then I should be able to settle this second issue.

I plan on attacking this problem in two steps. First I want to formulate a precise conjectural statement that we can "escape" from proper subfield type subfield subgroups of $\pi_P(\Gamma)$ for $P \in \mathbb{F}_p[t]$ irreducible and coprime to Q_0, and use this statement to prove that we can "escape" from these diagonal subgroups of $\pi_Q(\Gamma)$ for Q square free, coprime to Q_0. Once this has been shown, I will then try to prove the conjecture on "escaping" from subfield type subgroups. In order to do this I need to learn more aout the "small lifts" of elements in a subfield type subgroup.

After this I would like to try to get a similar result when the Zariski-closure of Γ is semisimple.

Question 2. *Let $p \geq 5$ be a prime number, $\Omega \subset \mathrm{GL}_n(\mathbb{F}_p[t, 1/Q_0])$ be a finite symmetric set, $\Gamma = \langle \Omega \rangle$, and $\mathbb{G}$ be the Zariski-closure of Γ. Suppose $\mathbb{G}$ is semisimple and simply connected and let $pr_i : \mathbb{G} \to \mathbb{G}_i$ be the projection of $\mathbb{G}$ onto its i^{th} almost simple factor. Assume that $\mathrm{pr}_i(\Gamma)$ is Zariski dense for all i and that the ring generated by $\mathrm{Tr}(\mathrm{pr}_i(\mathrm{Ad}\,\Gamma)))$ is all of $\mathbb{F}_p[t, 1/Q_0]$. Then is the family of graphs*

$$\{\mathrm{Cay}(\pi_Q(\Gamma), \pi_Q(\Omega))\}_{Q \in \mathbb{F}_p[t] \text{ square free, } (Q,Q_0)=1}$$

a family of ε-expanders for some $\varepsilon > 0$?

A large portion of our proof still works in this setting so I suspect this question of reasonable.

Aside from these questions, it may also be reasonable to answer the analogous positive characteristic version of the work of [6]. Namely, can we take the reduction mod P^n map were P is a fixed irreducible polynomial in $\mathbb{F}_p[t]$ and n ranges through the positive integers. A modest start would be the following

Question 3. *Let Ω be a finite symmetric set of $\mathrm{SL}_2(\mathbb{F}_p[t])$ which generates a Zariski-dense subgroup Γ. Then is the family of graphs*

$$\{\mathrm{Cay}(\pi_{t^n}(\Gamma), \pi_{t^n}(\Omega))\}_{n\geq 1}$$

a family of ε-expander graphs for some $\varepsilon > 0$?

Applications of expansion in positive characteristic. As I mentioned in the introduction, "superstrong approximation" in characteristic zero has found many applications. I would like to explore whether or not these applications have positive characteristic analogs. One immediate application is Lubotzky and Meiri's "Large Group Sieve" [14]. I would like to modify their proof to solve the following

Question 4. *Let Γ be a finitely generated subgroup of $\mathrm{GL}_n(\overline{\mathbb{F}_p(t)})$ which is not virtually-solvable. Then is the set of proper powers $\cup_{m\geq 2, p\nmid m}\Gamma^m$ is exponentially small in Γ?*

Here, *exponentially small* means that if Ω is any generating set of Γ then the probability that a word of length k in the alphabet Ω is exponentially small in k. A positive solution to Question 1 will imply a positive solution to Question 4 with a bit of work. Similarly by using the "Large Group Sieve", a positive solution to Question 1 would lead to a positive analog to the main theorem in [15].

References

[1] N. Alon and V. D. Milman. λ_1, isoperimetric inequalities for graphs, and superconcentrators. *J. Combin. Theory Ser. B*, 38(1):73–88, 1985.

[2] Jean Bourgain and Alex Gamburd. Expansion and random walks in $\mathrm{SL}_d(\mathbb{Z}/p^n\mathbb{Z})$. I. *J. Eur. Math. Soc. (JEMS)*, 10(4):987–1011, 2008.

[3] Jean Bourgain and Alex Gamburd. Uniform expansion bounds for Cayley graphs of $\mathrm{SL}_2(\mathbb{F}_p)$. *Ann. of Math. (2)*, 167(2):625–642, 2008.

[4] Jean Bourgain and Alex Gamburd. Expansion and random walks in $\mathrm{SL}_d(\mathbb{Z}/p^n\mathbb{Z})$. II. *J. Eur. Math. Soc. (JEMS)*, 11(5):1057–1103, 2009. With an appendix by Bourgain.

[5] Jean Bourgain, Alex Gamburd, and Peter Sarnak. Sieving and expanders. *C. R. Math. Acad. Sci. Paris*, 343(3):155–159, 2006.

[6] Jean Bourgain, Alex Gamburd, and Peter Sarnak. Affine linear sieve, expanders, and sum-product. *Invent. Math.*, 179(3):559–644, 2010.

[7] Emmanuel Breuillard, Ben Green, and Terence Tao. The structure of approximate groups. *Publ. Math. Inst. Hautes Études Sci.*, 116:115–221, 2012.

[8] Emmanuel Breuillard and Hee Oh, editors. *Thin groups and superstrong approximation*, volume 61 of *Mathematical Sciences Research Institute Publications*. Cambridge University Press, Cambridge, 2014. Selected expanded papers from the workshop held in Berkeley, CA, February 6–10, 2012.

[9] W. T. Gowers. Quasirandom groups. *Combin. Probab. Comput.*, 17(3):363–387, 2008.

[10] Jens Carsten Jantzen. *Representations of algebraic groups*, volume 107 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2003.

[11] D. Kazhdan. On the connection of the dual space of a group with the structure of its closed subgroups. 1:63–65, 1967.

[12] Michael J. Larsen and Richard Pink. Finite subgroups of algebraic groups. *J. Amer. Math. Soc.*, 24(4):1105–1158, 2011.

[13] Martin W. Liebeck and Gary M. Seitz. The maximal subgroups of positive dimension in exceptional algebraic groups. *Mem. Amer. Math. Soc.*, 169(802):vi+227, 2004.

[14] Alexander Lubotzky and Chen Meiri. Sieve methods in group theory I: Powers in linear groups. *J. Amer. Math. Soc.*, 25(4):1119–1148, 2012.

[15] Alexander Lubotzky and Lior Rosenzweig. The Galois group of random elements of linear groups. *Amer. J. Math.*, 136(5):1347–1383, 2014.

[16] G. A. Margulis. Explicit group-theoretic constructions of combinatorial schemes and their applications in the construction of expanders and concentrators. *Problemy Peredachi Informatsii*, 24(1):51–60, 1988.

[17] Madhav V. Nori. On subgroups of $\mathrm{GL}_n(\mathbf{F}_p)$. *Invent. Math.*, 88(2):257–275, 1987.

[18] Vladimir Platonov and Andrei Rapinchuk. *Algebraic groups and number theory*, volume 139 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.

[19] L. Pyber and E. Szabó. Growth in finite simple groups of lie type of bounded rank.

[20] Alireza Salehi Golsefidy and Péter P. Varjú. Expansion in perfect groups. *Geom. Funct. Anal.*, 22(6):1832–1891, 2012.

[21] Peter Sarnak and Xiao Xi Xue. Bounds for multiplicities of automorphic representations. *Duke Math. J.*, 64(1):207–227, 1991.

[22] G. B. Seligman. *Modular Lie algebras.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 40. Springer-Verlag New York, Inc., New York, 1967.

[23] Péter P. Varjú. Expansion in $SL_d(\mathcal{O}_K/I)$, I square-free. *J. Eur. Math. Soc. (JEMS)*, 14(1):273–305, 2012.

[24] Boris Weisfeiler. Strong approximation for Zariski-dense subgroups of semisimple algebraic groups. *Ann. of Math. (2)*, 120(2):271–315, 1984.