Longo: Math 20C - Winter 2017
Lecture Notes

Date: January 9, 2017

Section:

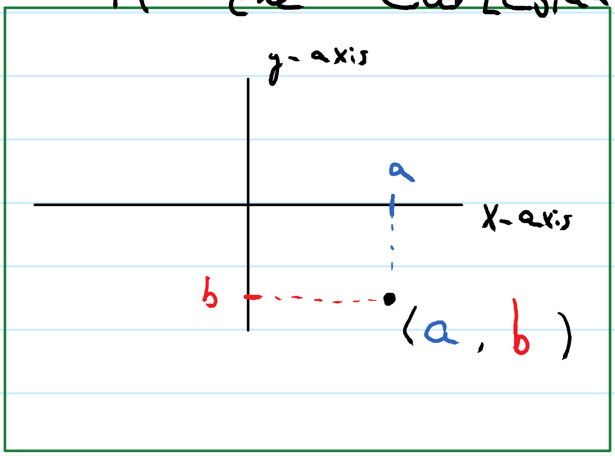
- [ ] §1.1

Topics Covered:

- [ ] 2 - Space and 3 - Space / basic arithmetic

§1.1: Vectors and basic operations:

So far, you've dealt mostly with points (a, b) in the Cartesian plane:



From now on, we will denote the set of all points (a, b), where a and are real numbers by $\mathbb{R}^2$ ("2 dimensional real space," or simply "2 space").

In calculus so far, you have been working with functions

$$f: \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto y = f(x)$$

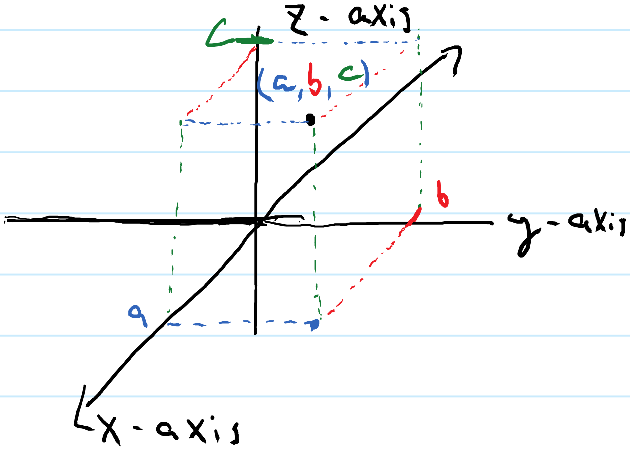
that take one real number 'x' as input, and return one real number 'y' as an output. In this class, we will try to do calculus on functions

$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x, y) \longmapsto z = f(x, y)$$

that take in a point '(x, y)' on the plane and spit out a real number 'z'.

Before we get there, we need to discuss geometry in 3 dimensional space in detail.

"3-space": Any triple (a, b, c) where a, b, c are real numbers represents a point in three dimensional space



In this picture, imagine the xy-plane is flat on the ground, and the z-axis is straight up out of the floor.

Remarks: ① The set of all such triples is denoted $\mathbb{R}^3$. ("3-space" or 3 dimentional real space).

② If (a, b, c) is a point in $\mathbb{R}^3$.

a is called the "x-component"

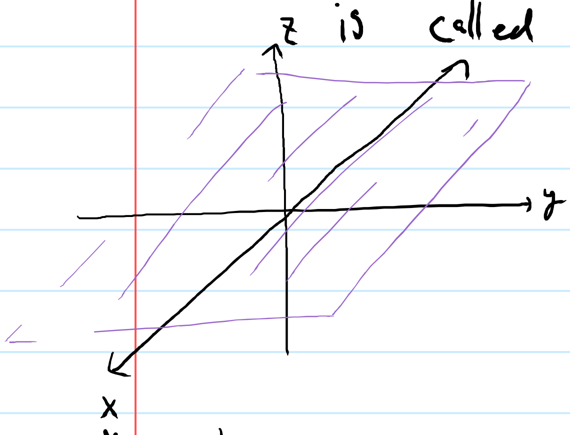
b is called the "y-component"

c is called the "z-component"

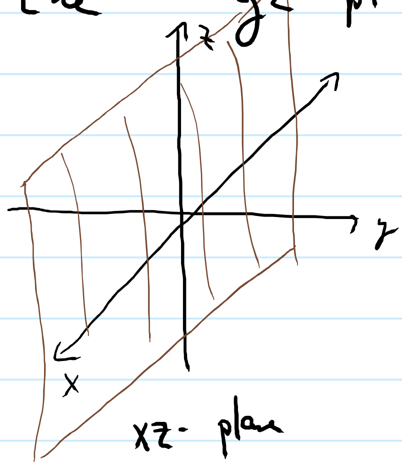
③ • The set of all points of the form $(a, b, 0)$ is called the "xy-plane"

• The set of all points of the form $(a, 0, c)$ is called the "xz-plane".

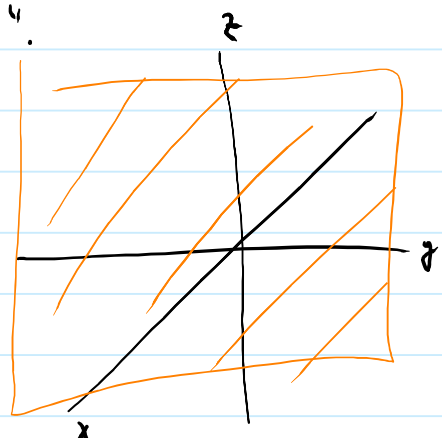
• The set of all points of the form $(0, b, c)$ is called the "yz-plane".



xy-plane



xz-plane



yz-plane.

Note: Planes are insanely hard to draw... sorry.

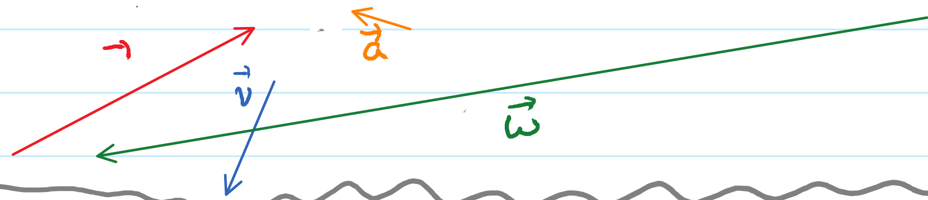
If we want to do Calculus on multivariable functions, we had better talk about how to do arithmetic using points in 2 or 3 space.

To do this, let's introduce the notion of

Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$: For our purposes, a Vector in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) is just a directed line segment in 2-space (resp. 3-space).

Since it is easier to draw in $\mathbb{R}^2$, leti start there.

Ex:

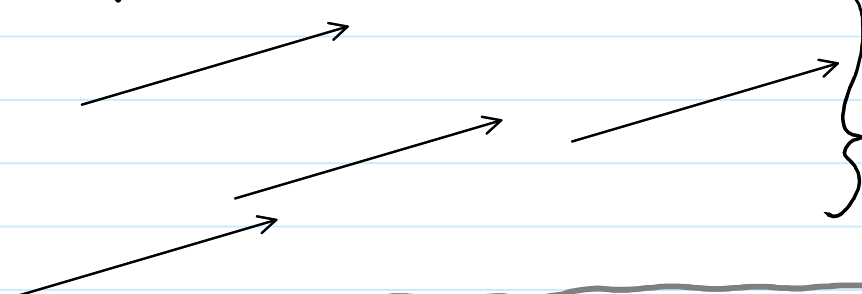


Remark: ① Vectors are usually written as a letter with an arrow above it: '$\vec{v}$'.

② A vector is determined by its length and direction. We don't really care where a vector starts or stops, what we are interated in is the relative change in position

With remark ② in mind, we call two vectors $\vec{u}$, $\vec{v}$ equivalent if they have the same direction and length.

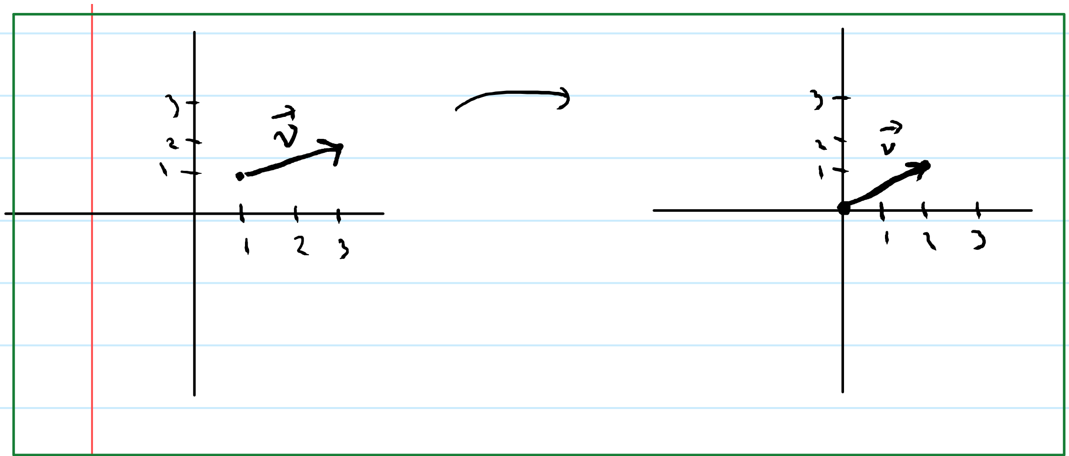
Ex:



all equivalent vectors.

How do we express vectors algebraically?

Given any vector, $\vec{v}$, $\mathbb{R}^2$ (or $\mathbb{R}^3$!), we can translate $\vec{v}$ so that its tail is at the origin.



Now the head of $\vec{v}$ stops at the point $(2,1)$. This means the vector goes $+2$ in the x-direction and $+1$ in the y-direction. By an abuse of notation, we say

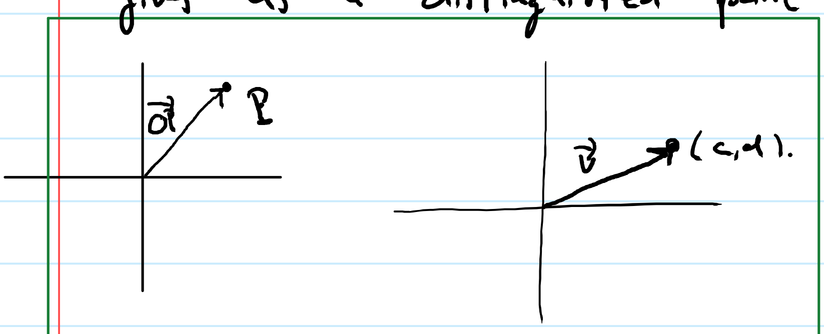
$$\boxed{\vec{v} = (2,1).}$$

Remark: Here, 2 is called the "x-component"
1 is called the "y-component"

Notice that this is the same notation we use for points in $\mathbb{R}^2$. We can do this because

Points and vectors starting at the origin are essentially the same thing

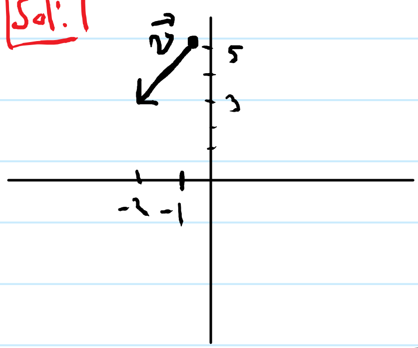
More Precisely, given a point $P=(a,b)$ on the plane, we can construct a vector starting at the origin by drawing a line from the origin to P. We call this vector $\vec{OP}$. Conversely if we have a vector $\vec{v} = (c, d)$ starting at the origin, its "head", (c,d) gives us a distinguished point in the plane.



We will often blur the notion of points and vectors into a single concept.

Example: Find the components of the vector that starts at $(-1,3)$ and ends at $(-2,5)$.

Sol:



Since $\vec{v}$ moves -1 in the x-direction and -2 in the y-direction.

$\vec{v} = (-1,-2)$

More generally the vector from $P=(x_1,y_1)$ to $Q=(x_2,y_2)$ is $(x_2-x_1,\ y_2-y_1)$ and is denoted $\overrightarrow{PQ}$

Vector addition and scalar multiplication:

Remark: For us, "scalar" just means real number.

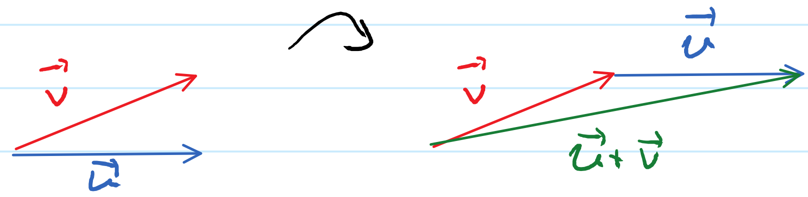
Given any two vectors $\vec{u}=(x_1,y_1)$, $\vec{v}=(x_2,y_2)$ and any real number k, we can add the vectors and multiply by the scalar k by doing the operations componentwise.

1. Vector addition: $\vec{u}+\vec{v} = (x_1,y_1)+(x_2,y_2) = (x_1+x_2,\ y_1+y_2)$
2. scalar multiplication: $k\vec{u} = k(x_1,y_1) = (kx_1,\ ky_2)$

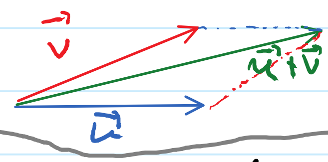
Rmk: Vector addition and scalar multiplication have nice properties such as commutativity, associativity, and distribution. (See book for details)

What does this mean geometrically?

① Given two vectors, $\vec{u}, \vec{v}$, draw them so that they start at a common point.

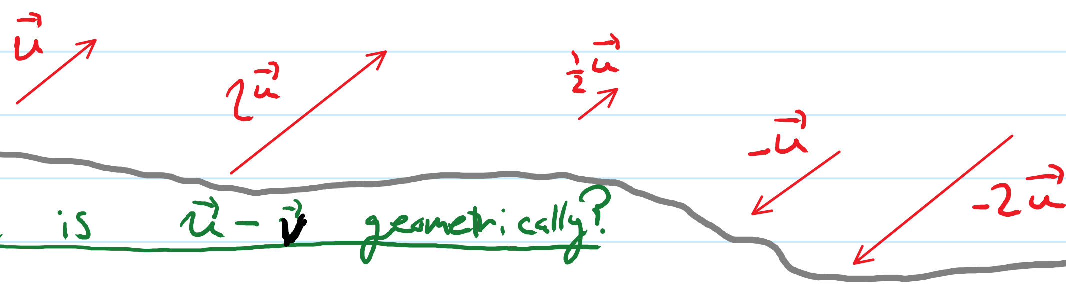


move $\vec{u}$ so that its tail is on the head of $\vec{v}$. $\vec{u}+\vec{v}$ is the vector that starts at the base of $\vec{v}$ and ends at the head of $\vec{u}$. Equivalently, use the parallelogram rule:



② If $k>0$, then $k\vec{u}$ is just what you get if you stretch $\vec{u}$ by a factor of k. If $k<0$, the resulting vector points the opposite direction, as well as being stretched.

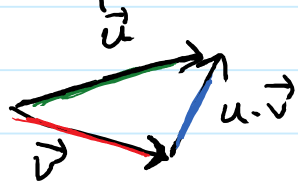
Ex:



What is $\vec{u}-\vec{v}$ geometrically?

Notice the vector $\vec{u}-\vec{v}$ is what you add to $\vec{v}$ in order to get $\vec{u}$: $\boxed{\vec{v}+(\vec{u}-\vec{v})=\vec{u}}$

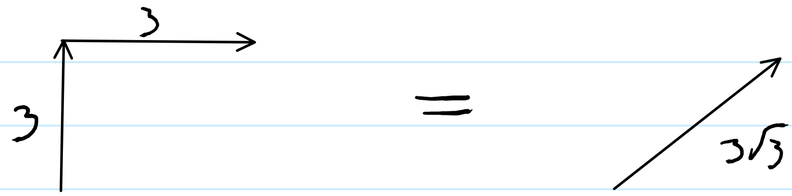
So we have the picture:



So $\vec{u} - \vec{v}$ goes from the head of $\vec{v}$ to the head of $\vec{u}$.

Remark: You should think of vector addition, $\vec{u}+\vec{v}$ as moving along $\vec{u}$ and then moving along $\vec{v}$.

For example: "Go North 3 units" + "Go East 3 units" is the same as "Go North-East $3\sqrt{2}$ units".



Next time:
1. Length of vectors / normalization
2. lines.
3. inner product.