Longo: Math 20C - Winter 2017
Lecture Notes

Date:

Section:

- [ ] §2.4

Topics Covered:

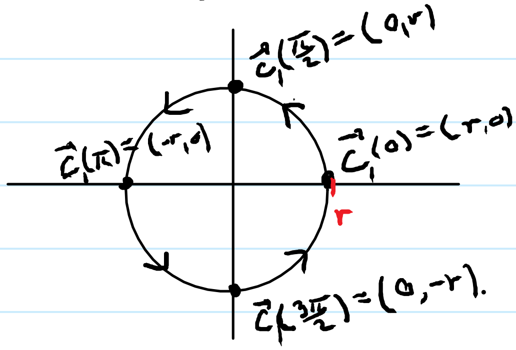
- [ ] Paths and curves

§2.4: Paths and Curves:

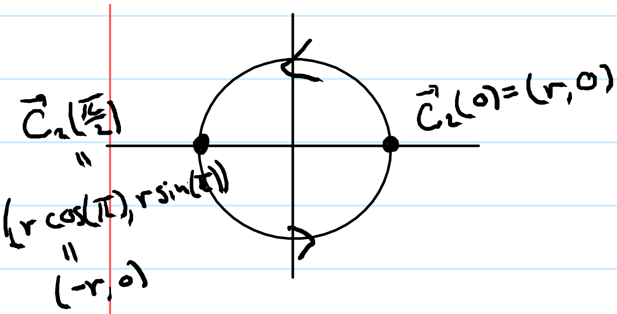
Today we consider the geometry of fcns $\vec{c}:\mathbb{R}\to\mathbb{R}^2$ (or $\mathbb{R}^3$).

For these fcns, we think of the input variable, t, as time. Then for any time value, t_0, $\vec{c}(t_0)$ tells us the position of a particle floating in space. As time progresses, the particle leaves a "dust trail" in its wake. This dust trail will be some curve, C, in 2-space (or 3-space). We say, $\vec{c}$ parametrizes the curve C.

Examples: ① The circle of radius r is given by the equation: $x^2+y^2=r^2$. It can be parametrized by the the fcn
$$\vec{c}_1:\mathbb{R}\to\mathbb{R}^2,\quad \vec{c}_1(t)=(r\cos(t), r\sin(t)).$$
The point $\vec{c}_1(0)=(r\cos(0), r\sin(0))=(r,0)$ is the intersection of the curve with the positive x-axis. As t increases, $\vec{c}(t)$ travels counterclockwise around the circle.



② Let $\vec{c}_2:\mathbb{R}\to\mathbb{R}^2$, $\vec{c}_2(t)=(r\cos(2t), r\sin(2t))$. $\vec{c}_2(t)$ also traces the circle, but it travels around the circle twice as fast:



This example shows that two different functions can parametrize the same curve.

③ $\vec{c}_3 : \mathbb{R} \to \mathbb{R}^3$, $\vec{c}_3(t) = (\cos(t), \sin(t), t)$.

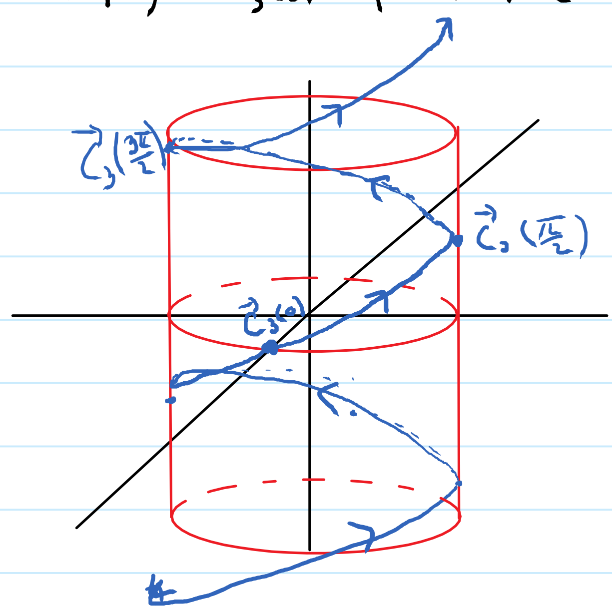
The x and y coordinates always satisfy the equation

$$x^2 + y^2 = 1.$$

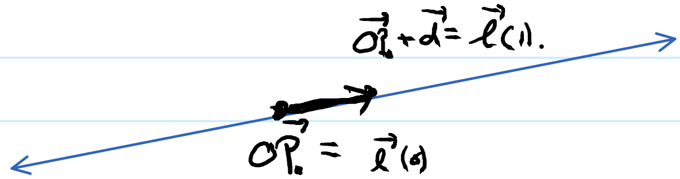
So $\vec{c}_3(t)$ always lies on the cylinder $x^2+y^2=1$

(Note: $x^2+y^2=1$ defines a cylinder in 3-space because the z-coordinate can be any number)

Meanwhile, the z-coordinate rises as time moves on. Therefore, $\vec{c}_3(t)$ parametrizes a helix.



④ If P_0 is a point in $\mathbb{R}^3$, $\vec{d}$ is a direction vector, we've already seen the parametrization of the line $\vec{\ell}(t) = \overrightarrow{OP_0} + t\vec{d}$



Velocity vectors:

Since we think of $\vec{c}(t)$ as the position of some particle floating in space, it is reasonable to ask about the speed and direction in which the particle is traveling. We do this with the velocity vector.

Def: If $\vec{c}(t)$ is a path, then the velocity of $\vec{c}$ at time t is defined by

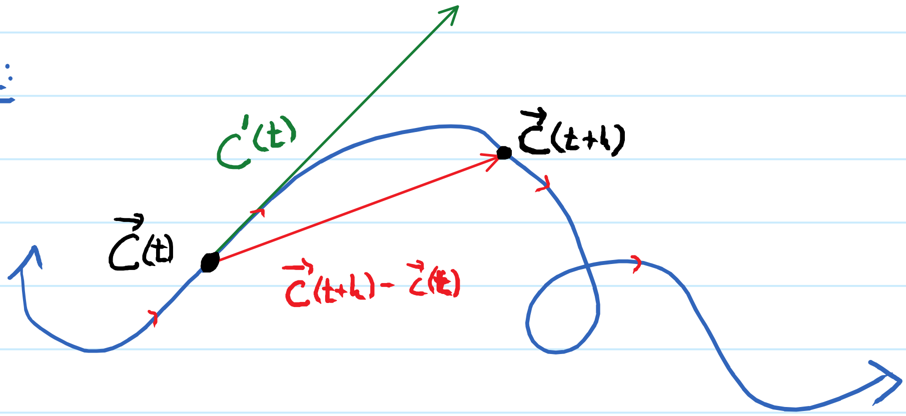
$$\boxed{v(t) = \vec{c}\,'(t) = \lim_{h \to 0} \frac{\vec{c}(t+h) - \vec{c}(t)}{h}}$$

The speed of $\vec{c}$, denoted $s(t)$ is defined to be

$$\boxed{s(t) = \|\vec{c}\,'(t)\|}$$

It is not hard to see that: ① if $\vec{c}(t) = (x(t), y(t))$, then $\boxed{\vec{c}\,'(t) = (x'(t), y'(t))}$. ② If $\vec{c}(t) = (x(t), y(t), z(t))$, then $\boxed{\vec{c}\,'(t) = (x'(t), y'(t), z'(t))}$.

Picture:



Rmk: ① If we write $\vec{c}:\mathbb{R}\to\mathbb{R}^3$. Then (from §2.3) we have

$$[D\vec{c}](t)=\begin{bmatrix}\frac{dx}{dt}(t)\\ \frac{dy}{dt}(t)\\ \frac{dz}{dt}(t)\end{bmatrix}$$

Which is just $\vec{c}\,'(t)$ but written as a column vector. So this notion of derivative is consistent with the other definition. We like to write $\vec{c}\,'(t)$ as a row vector because we want to view $\vec{c}\,'(t)$ as a vector in space.

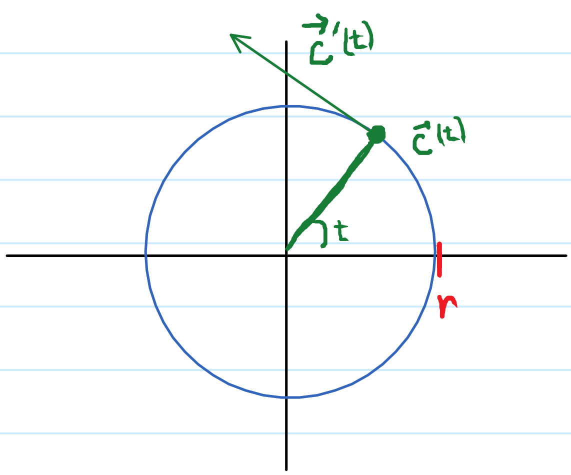
② It is ok to use the "prime" notation here because there is only one input variable, so there is no ambiguity.

Examples: ① $\vec{c}_1(t)=(r\cos(t), r\sin(t))$ for some constant r.

$$\vec{c}_1\,'(t)=(-r\sin(t), r\cos(t)).$$

$$\begin{aligned} s(t)=\|\vec{c}_1\,'(t)\| &= \left((-r\sin(t))^2+(r\cos(t))^2\right)^{1/2} \\ &= \left(r^2\sin^2(t)+r^2\cos^2(t)\right)^{1/2} \\ &= (r^2)^{1/2} \\ &= r \end{aligned}$$

(constant speed).

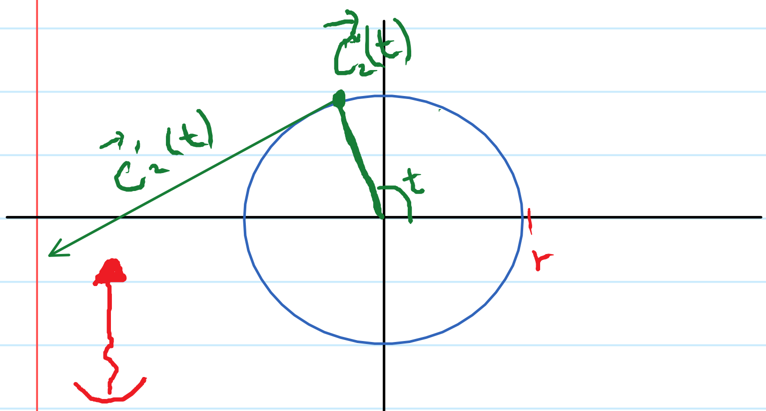


② $\vec{c}_2(t) = (r\cos(2t), r(\sin(2t))$

$$c_2'(t) = \left(-2r\sin(2t),\ 2r\cos(2t)\right)$$

$$s_2(t) = \|c_2'(t)\| = \left((-2r\sin(2t))^2 + (2r\cos(2t))^2\right)^{1/2}$$
$$= \left(4r^2\sin^2(2t) + 4r^2\cos^2(2t)\right)^{1/2}$$
$$= (4r^2)^{1/2}$$
$$= 2r$$

(constant speed but twice as fast!)

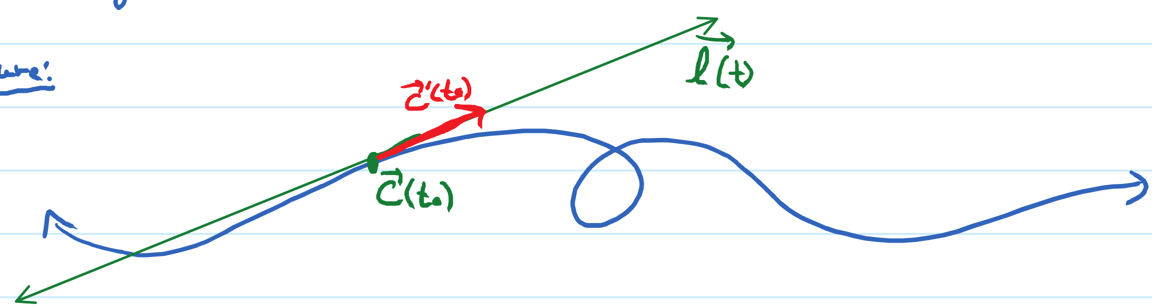


velocity vector is twice as long as $\vec{c}_1\,'(t)$

Tangent Lines to Paths:

Given a path $\vec{c}(t)$, we can ask for the equation of the line tangent to the C (the curve traced out by $\vec{c}(t)$), at a specific time $t = t_0$.

Picture:



Geometrically, if we imagine the particle "detaching" from the curve C at time t_0, it will fly off and continue along the line $\vec{\ell}(t)$.

Luckily, we already have a point on the line: $\vec{c}(t_0)$, and a direction vector: $\vec{c}\,'(t_0)$. So we could say

$$\vec{\ell}(t) = \vec{c}(t_0) + t\,\vec{c}\,'(t_0).$$

However, we prefer to "shift" the formula so that $\vec{\ell}(t_0) = \vec{c}(t_0)$ (so that we imagine the particle flying off the track at time $t = t_0$).

So the equation we will use is

$$\boxed{\vec{\ell}(t) = \vec{c}(t_0) + (t - t_0)\,\vec{c}\,'(t_0)}$$

<u>Note</u>: Compare this to the linear approximation equation! It's the same thing again.

<u>Example</u>: Find the equation of the line that is tangent to the path $\vec{c}(t) = (t\cos(t), t\sin(t), t)$ at $t = \pi$.

<u>Sol</u>: Calculate: ① $\vec{c}(\pi) = (\pi\cos(\pi), \pi\sin(\pi), \pi)$
$= (-\pi, 0, \pi)$

② $\vec{c}\,'(t) = (\cos(t) - t\sin(t), \sin(t) + t\cos(t), 1)$
$\vec{c}\,'(\pi) = (\cos(\pi) - \pi\sin(\pi), \sin(\pi) + \pi\cos(\pi), 1)$
$= (-1, -\pi, 1)$. So

$$\boxed{\vec{\ell}(t) = (-\pi, 0, \pi) + (t - t_0)(-1, -\pi, 1).}$$