Longo: Math 20C - Winter 2017
Lecture Notes

Date:

Section:
- ☐ § 2.4 (cont.)
- ☐ § 2.5

Topics Covered:
- ☐ Equation of tangent lines to paths
- ☐ First Properties of the derivative

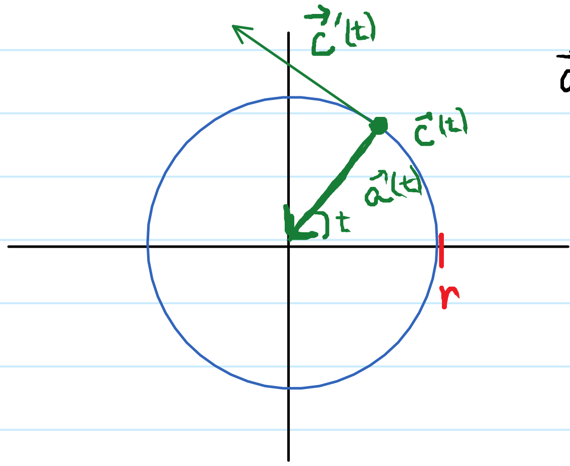
From last Time: If $\vec{c}:\mathbb{R}\to\mathbb{R}^2$ or $\mathbb{R}^3$ is a path. The velocity vector at time t is given by $\vec{v}(t) = \vec{c}\,'(t) = (x'(t), y'(t))$ (or $(x'(t), y'(t), z'(t))$), and the speed of $\vec{c}$ at time t is given by $s(t) = \|\vec{c}\,'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}$ (or $\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}$).

We can go one step further, and define the acceleration vector at time t to be $\vec{a}(t) = \vec{v}\,'(t) = \vec{c}\,''(t)$. The acceleration vector tells you in which direction the velocity is changing.

Examples: ① $\vec{c}_1(t) = (r\cos(t), r\sin(t))$ for some constant r.

$$\vec{c}_1\,'(t) = (-r\sin(t), r\cos(t)).$$

$$\begin{aligned} s(t) = \|\vec{c}_1\,'(t)\| &= \left((-r\sin(t))^2 + (r\cos(t))^2\right)^{1/2} \\ &= \left(r^2\sin^2(t) + r^2\cos^2(t)\right)^{1/2} \\ &= (r^2)^{1/2} \\ &= r \quad \text{(constant speed).} \end{aligned}$$



$$\vec{a}(t) = \vec{c}_1\,''(t) = (-r\cos(t), -r\sin(t))$$

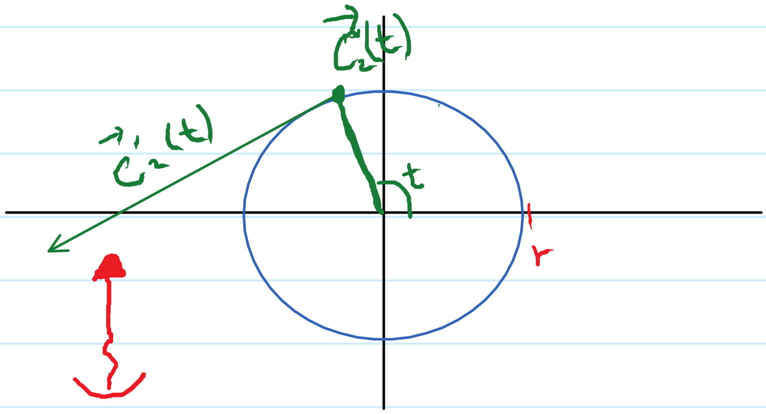
② $\vec{c}_2(t) = (r\cos(2t), r(\sin(2t))$

$$c_2'(t) = (-2r\sin(2t), 2r\cos(2t))$$

$$\begin{aligned} s_2(t) = \|c_2'(t)\| &= \left((-2r\sin(2t))^2 + (2r\cos(2t))^2\right)^{1/2} \\ &= \left(4r^2\sin^2(2t) + 4r^2\cos^2(2t)\right)^{1/2} \end{aligned}$$

$$= (4r^2)^{1/2}$$
$$= 2r$$

(constant speed but twice as fast!)

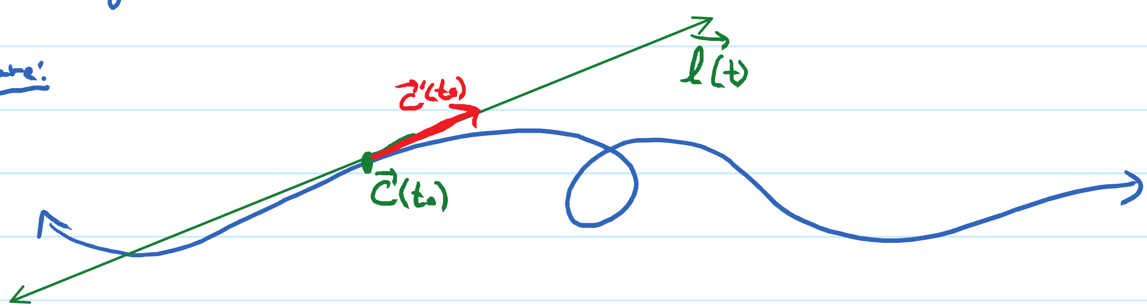


velocity vector is twice as long as $\vec{c}_1\,'(t)$

Tangent Lines to Paths:

Given a path $\vec{c}(t)$, we can ask for the equation of the line tangent to the C (the curve traced out by $\vec{c}(t)$), at a specific time $t = t_0$.

Picture:



Geometrically, if we imagine the particle "detaching" from the curve C at time t_0, it will fly off and continue along the line $\vec{\ell}(t)$.

Luckily, we already have a point on the line: $\vec{c}(t_0)$, and a direction vector: $\vec{c}\,'(t_0)$. So we could say

$$\vec{\ell}(t) = \vec{c}(t_0) + t\,\vec{c}\,'(t_0).$$

However, we prefer to "shift" the formula so that $\vec{\ell}(t_0) = \vec{c}(t_0)$ (so that we imagine the particle flying off the track at time $t = t_0$).

So the equation we will use is

$$\boxed{\vec{\ell}(t) = \vec{c}(t_0) + (t - t_0)\,\vec{c}\,'(t_0)}$$

Note: Compare this to the linear approximation equation! It's the same thing again.

Example: Find the equation of the line that is tangent to the path $\vec{c}(t) = (t\cos(t), t\sin(t), t)$ at $t = \pi$.

Sol: Calculate: ① $\vec{c}(\pi) = (\pi\cos(\pi), \pi\sin(\pi), \pi)$
$= (-\pi, 0, \pi)$

② $\vec{c}\,'(t) = (\cos(t) - t\sin(t), \sin(t) + t\cos(t), 1)$
$\vec{c}\,'(\pi) = (\cos(\pi) - \pi\sin(\pi), \sin(\pi) + \pi\cos(\pi), 1)$
$= (-1, -\pi, 1)$. so

$$\boxed{\vec{\ell}(t) = (-\pi, 0, \pi) + (t - t_0)(-1, -\pi, 1).}$$

§2.5: Properties of the derivative (with an emphasis on the chain rule):

The main objective in this section will be to explain the multivariable versions of the product rule and the chain rule. Since we discussed the total derivative of a multivariable fcn, the chain rule will look very familiar.

Basic Properties of derivatives:

Let's quickly review some basic operations involving matrices:

① It makes sense to multiply a matrix by a scalar:

Example: $2\begin{bmatrix} 0 & 3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ -2 & 8 \end{bmatrix}$

② If A, B are $n\times m$ matrices (A, B are the same dimension) you can add matrices componentwise:

Example: $\begin{bmatrix} 2 & 0 \\ 1 & 6 \\ -4 & 10 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 20 & 11 \\ 5 & -6 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 21 & 17 \\ 1 & 4 \end{bmatrix}$

③ If A is an $n\times m$ matrix and B is an $m\times p$ matrix, then we can multiply AB. The entry of AB in the i^{th} row and j^{th} column is the dot product of the i^{th} row of A with the j^{th} row of B. The result is an $n\times p$ matrix.

Example:

$$2\times 3\left\{\begin{bmatrix} 1 & -1 & 0 \\ 5 & 7 & -9 \end{bmatrix}\right. \underbrace{\begin{bmatrix} 8 & 8 \\ 0 & 3 \\ -2 & 1 \end{bmatrix}}_{3\times 2} = \begin{bmatrix} 8 & 5 \\ 58 & 31 \end{bmatrix}.$$

Theorem: (Basic Properties)

① If $f: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is diff'ble at $\vec{x}_0$, and c is a scalar, $[D(cf)](\vec{x}_0) = c\,[D(f)](\vec{x}_0)$.

(We can pull out scalars).

(2) If $f, g: \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ are diff'ble at $\vec{x}_0$, then $f+g: \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

$$[D(f+g)](\vec{x}_0) = [Df](\vec{x}_0) + [Dg](\vec{x}_0)$$

These are all $n \times m$ matrices.

(3) (Product Rules) (i) If $f, g: \mathbb{R}^n \to \mathbb{R}$, then $fg: \mathbb{R}^n \to \mathbb{R}$.
If f, g are diff'ble at $\vec{x}_0$

$$\nabla(fg)(\vec{x}_0) = f(\vec{x}_0)\nabla g(\vec{x}_0) + g(\vec{x}_0)\nabla f(\vec{x}_0)$$

Scalar mult. ($\nabla f, \nabla g$ are n-dim. vectors!)

(ii) If $\vec{c}(t)$, $\vec{r}(t)$ are paths, then

$$\frac{d}{dt}(\vec{c}(t) \cdot \vec{r}(t)) = \vec{c}\,'(t) \cdot \vec{r}(t) + \vec{c}(t) \cdot \vec{r}\,'(t)$$
$$\frac{d}{dt}(\vec{c}(t) \times \vec{r}(t)) = (\vec{c}\,'(t) \times \vec{r}(t)) + (\vec{c}(t) \times \vec{r}\,'(t)).$$

Example: Consider the parametrization of the circle $x^2 + y^2 = r^2$ from before:

$\vec{c}(t) = (r\cos(t), r\sin(t))$. We observed before that: (1) $\vec{c}$ had constant speed r. I.e. $s(t) = \|\vec{c}\,'(t)\| = r$ for all t.

(2) The accelleration vector $\vec{a}(t)$ appeared to be orthogonal to velocity $\vec{v}(t)$.

Let's prove this is indeed true:

Since speed is constant, $\|\vec{c}'(t)\| = r$ is constant, and therefore, $\|\vec{c}'(t)\|^2 = \vec{c}'(t)\cdot\vec{c}'(t) = r$ is constant.

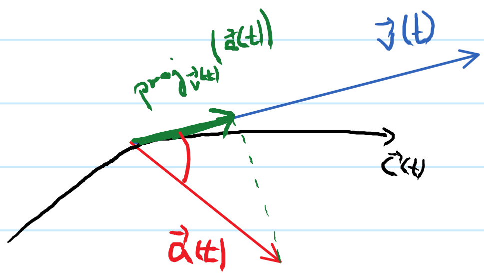
Taking derivatives on both sides, we get

$$\frac{d}{dt}\left(\vec{c}'(t)\cdot\vec{c}'(t)\right) = \frac{d}{dt}(r) = 0$$

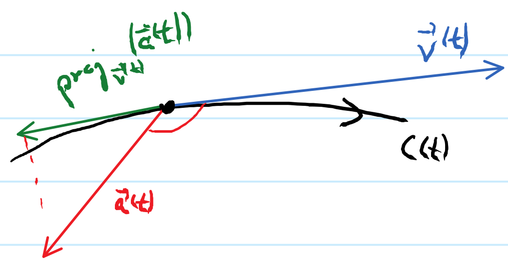
(derivative of a constant is 0.)

$$\Rightarrow \quad \vec{c}''(t)\cdot\vec{c}'(t) + \vec{c}'(t)\cdot\vec{c}''(t) = 0$$
$$\Rightarrow \quad 2\left(\vec{c}''(t)\cdot\vec{c}'(t)\right) = 0$$
$$\Rightarrow \quad \vec{c}''(t)\cdot\vec{c}'(t) = 0$$
$$\Rightarrow \quad \vec{a}(t)\cdot\vec{v}(t) = 0$$
$$\Rightarrow \quad \vec{a}(t) \perp v(t) \quad \text{for all } t.$$

Rmk: Intuitively, this makes sense. If the angle between $\vec{a}(t)$ and $\vec{v}(t)$ were acute, there would be some positive acceleration, which would cause the speed to increase. Similarly if the angle were obtuse, there would be some negative acceleration and the speed would decrease.



accelerating forward
$\Rightarrow$ speed increase.



accelerating backwards
$\Rightarrow$ speed decrease.