Longo: Math 20C - Winter 2017 Lecture Notes

Date:

Section:

- [ ] §2.5 (cont.)

Topics Covered:

- [ ] Chain rule for partial derivatives revisited.

Chain Rule for Partials: Last time we saw:

Theorem: (Chain Rule): Suppose $f: \mathbb{R}^m \to \mathbb{R}^p$, $g: \mathbb{R}^n \to \mathbb{R}^m$ are both diff'ble. Then

$$h := f \circ g : \mathbb{R}^n \to \mathbb{R}^p \text{ is also diff'ble}$$

and $$[Dh](\vec{x}) = [Df](g(\vec{x}))[Dg](\vec{x})$$

This means that we can obtain the full matrix of partial derivatives of $h = f \circ g$ by multiplying the matrices of partial derivatives of f and g. This is often more efficient than actually composing the functions and calculating the partials of h directly.

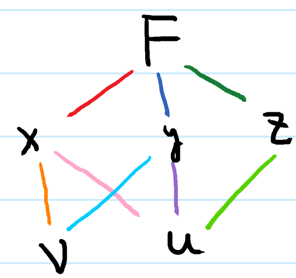
On the other hand, if we only need certain partials of h (and not the entire matrix $[Dh]$), we can do it in a quicker (and less confusing (?)) way.

Shortcut diagram for calculating partials of a composition:

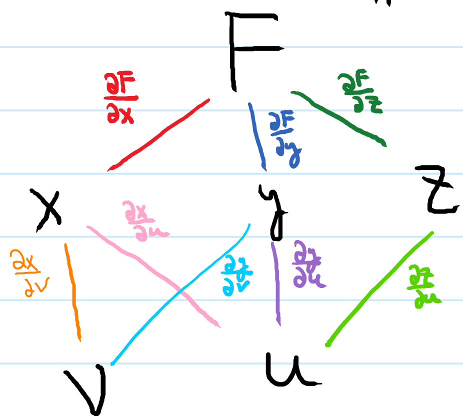
It is easiest to explain this via an example:

Suppose: ① $F(x,y,z)$ is a fcn of x, y, z,
② x, y are fcns of u, v
③ z is a fcn of only u.

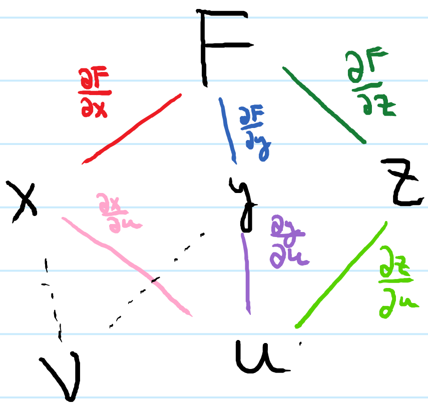
Then we can draw a dependency diagram



Here, we attach a line from one node to another if the higher node depends on the lower node. On this diagram, we attach the appropriate partial derivative:

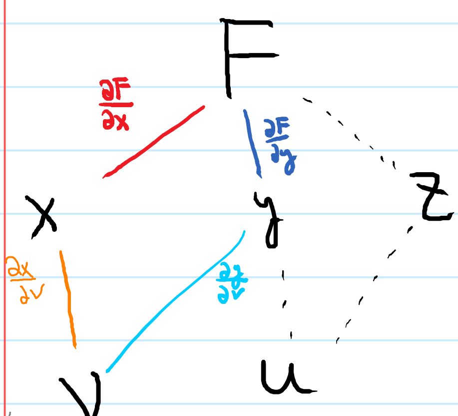


Now to find $\frac{\partial F}{\partial u}$, isolate 'u' in the tree and look at all paths from F to u. On each path, multiply the partials on the legs. Finally, add the paths together:



$$\boxed{\frac{\partial F}{\partial u} = \left(\frac{\partial F}{\partial x}\right)\left(\frac{\partial x}{\partial u}\right) + \left(\frac{\partial F}{\partial y}\right)\left(\frac{\partial y}{\partial u}\right) + \left(\frac{\partial F}{\partial z}\right)\left(\frac{\partial z}{\partial u}\right)}$$

If we do the same for v, we get:



$$\boxed{\frac{\partial F}{\partial v} = \left(\frac{\partial F}{\partial x}\right)\left(\frac{\partial x}{\partial v}\right) + \left(\frac{\partial F}{\partial y}\right)\left(\frac{\partial y}{\partial v}\right)}$$

Warning: To avoid messy notation, I didn't write where these fcns should be evaluated. Partials of f should be evaluated at $(x(u,v), y(u,v), z(u))$, partials of x, y, z, should be evaluated at (u,v).

Ex: In the example above, assume:

$x(1,2)=3$ $\quad \frac{\partial F}{\partial x}(3,5,6)=8$ $\quad \frac{\partial x}{\partial u}(1,2)=1$ $\quad \frac{\partial y}{\partial u}(1,2)=-1$

$y(1,2)=5$ $\quad \frac{\partial F}{\partial y}(3,5,6)=-1$ $\quad \frac{\partial x}{\partial v}(1,2)=2$ $\quad \frac{\partial y}{\partial u}(1,2)=5$

$z(1)=6$ $\quad \frac{\partial F}{\partial z}(3,5,6)=7$ $\quad \frac{\partial z}{\partial u}(1)=8$

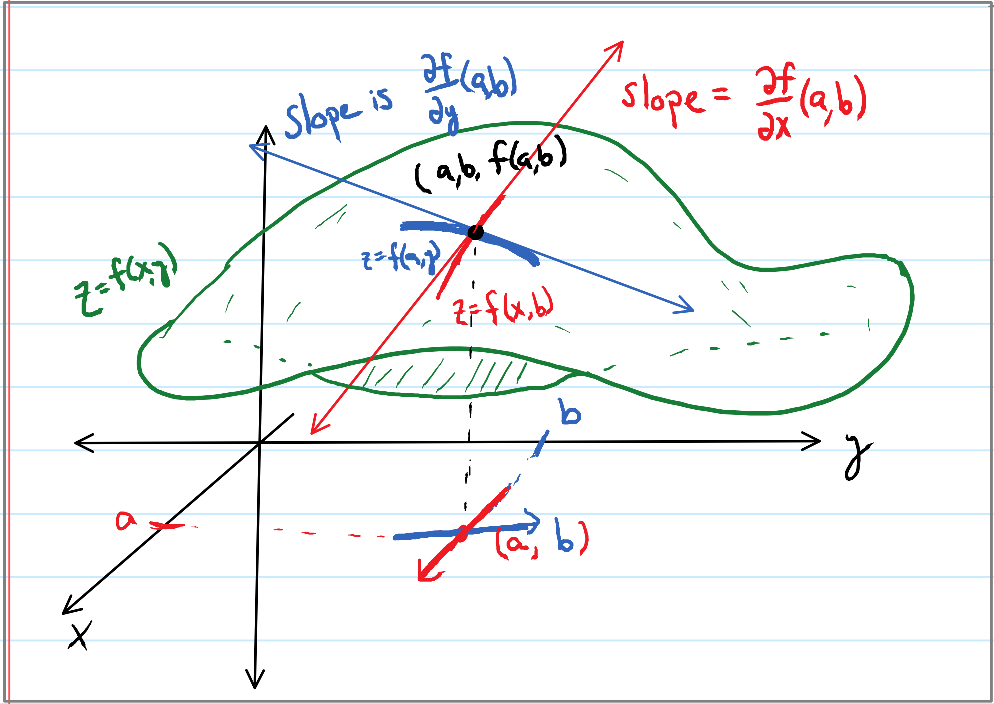
Then $\frac{\partial F}{\partial u}(1,2) = (3)(1)+(-1)(-1)+(7)(8)$
$= \boxed{60}$

Note: This notation means when $(u,v)=(1,2)$, $(x,y,z)=(3,5,6)$. So F's partials w.r.t. x,y,z should be evaluated at $(3,5,6)$.

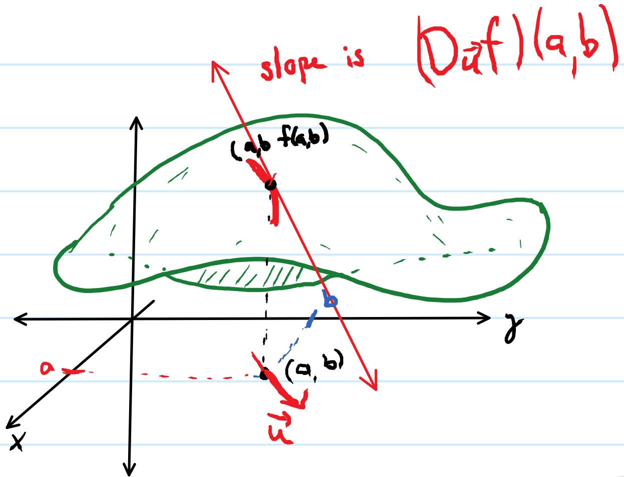
§2.6: Directional derivatives and the geometry of the gradient:

In this section, we stick to fcns from $\mathbb{R}^2\to\mathbb{R}$ or $\mathbb{R}^3\to\mathbb{R}$

Let $f:\mathbb{R}^2\to\mathbb{R}$. We saw that $\frac{\partial f}{\partial x}(a,b)$ (resp. $\frac{\partial f}{\partial y}(a,b)$) tell you the rate of change in f as we move from the point (a,b) in the positive x-direction (resp. positive y directn).



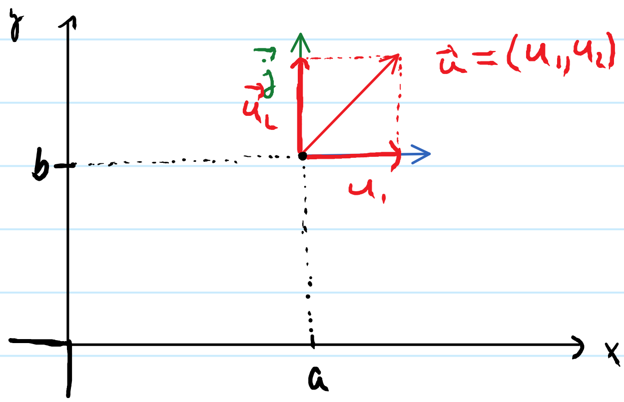
$\boxed{Q?}$ How do we determine the rate of change of f as we move from (a,b) in an arbitrary direction determined by a unit vector $\vec{u}$?



We will call this the directional derivative of f in the direction of $\vec{u}$, denoted $(D_{\vec{u}}f)(a,b)$

How do we calculate $(D_{\vec{u}}f)(a,b)$?

Look at the domain of f.



Decompose $\vec{u}$ into its components

$$\vec{u}=u_1\vec{i}+u_2\vec{j}$$

Since $\vec{u}$ "goes u_1 in the $\vec{i}$ and u_2 in the $\vec{j}$ direction, the total change in f as you move from (a,b) in the $\vec{u}$ direction is

$$(D_{\vec{u}}f)(a,b)=\underbrace{\frac{\partial f}{\partial x}(a,b)}_{\text{rate of change in } x \text{ dir.}}\underbrace{u_1}_{\text{distance travelled in } x \text{ dir.}}+\underbrace{\frac{\partial f}{\partial y}(a,b)}_{\text{rate of change in } y \text{ dir.}}\underbrace{u_2}_{\text{distance travelled in } y \text{ dir.}}$$

$$\boxed{=(\nabla f)(a,b)\cdot\vec{u}}$$

Example: Find the rate of change of

$$f(x,y,z)=2xy-e^{x^2y}+z^2 \quad \text{at} \quad (2,1,0)$$

in the direction of $\vec{u}=(1,-1,1)$.

1. According to my definition, we need $\vec{u}$ to be a unit vector, so we first normalize $\vec{u}$.
Replace $\vec{u}$ by $\vec{e}_{\vec{u}} = \frac{1}{\|\vec{u}\|}\vec{u} = \frac{1}{\sqrt{1+1+1}}(1,-1,1)$

$$= \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

2. The equation above was for fcns from $\mathbb{R}^2$ to $\mathbb{R}^1$, but the same formula works.

$$(\nabla f)(2,1,0) = \left(2y - 2xye^{x^2y},\ 2x - x^2e^{xy},\ 2z\right)\Big|_{(2,1,0)}$$

$$= (2-4e^4,\ 4-4e^4,\ 0)$$

$$(D_{\vec{u}}f)(2,1,0) = (2-4e^4,\ 4-4e^4,\ 0)\cdot\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}}(2-4e^4) - \frac{1}{\sqrt{3}}(4-4e^4)$$

$$= \boxed{\frac{-2}{\sqrt{3}}}$$

Differences with the book:

Warnings: 1. Your book also considers $D_{\vec{u}}f$ when $\vec{u}$ is not a unit vector. They call this the directional derivative along $\vec{u}$. We will only look at directional derivatives in the direction of $\vec{u}$.

2. The book uses the notation $[Df](a,b)(\vec{u})$ instead of $(D_{\vec{u}}f)(a,b)$.

This is because $(D_{\vec{u}}f)(a,b) = (\nabla f)(a,b)\cdot\vec{u}$

$$= [Df](a,b)\cdot\vec{u}$$

the total derivative of f

viewed as a column vector.

③ The book has a nice explanation as to what this means in higher dimension. We will ignore this, but I encourage you to read it since it may provide insight for the significance of $[Df](\vec{x})$.

Geometry of the gradient: If $\vec{u}$ is a unit vector,

$(D_{\vec{u}} f)(a,b) = (\nabla f)(a,b) \cdot \vec{u}$ is the rate of change in the $\vec{u}$ direction.

On the other hand:

$$(\nabla f)(a,b) \cdot \vec{u} = \|(\nabla f)(a,b)\| \cdot \|\vec{u}\| \cos\theta$$
$$= \|\nabla f(a,b)\| \cos\theta$$

where θ is the angle between $\vec{u}$ and $\nabla f(a,b)$.

Note: ① $(D_{\vec{u}} f)(a,b)$ is maximal when $\cos\theta = 1$

$\Leftrightarrow \theta = 0$

$\Leftrightarrow \vec{u}$ points in the same direction as $(\nabla f)(a,b)$

So $(\nabla f)(a,b)$ points in the direction of maximal increase for f.

② If $\vec{u} = \dfrac{\nabla f(a,b)}{\|\nabla f(a,b)\|}$ is the unit vector in the $\nabla f(a,b)$ direction,

$$(D_{\vec{u}} f)(a,b) = \nabla f(a,b) \cdot \left(\frac{\nabla f(a,b)}{\|\nabla f(a,b)\|} \right)$$
$$= \frac{\nabla f(a,b) \cdot \nabla f(a,b)}{\|\nabla f(a,b)\|}$$
$$= \frac{\|\nabla f(a,b)\|^2}{\|\nabla f(a,b)\|}$$

$$= \|\nabla f(a,b)\|$$

So the maximal rate of change of f in any direction is $\|\nabla f(a,b)\|$

Example: Farmer Corey is standing near the middle of his corn field in the great city of Omaha, NE (say he is standing at the point $(0,1)$). The temperature at the point (x,y) is given by $T(x,y) = 30e^{-(x-1)^2-(2y-1)^2}$. Farmer Corey is very cold! What direction should he walk to warm up the fastest?

Sol: He should walk in the direction of $(\nabla T)(0,1)$.

$$\nabla T = \left(-60(x-1)e^{-(x-1)^2-(2y-1)^2}, -120(2y-1)e^{-(x-1)^2-(2y-1)^2} \right)$$

$$(\nabla T)(0,1) = \left(60e^{-2}, -120e^{-2} \right).$$