Longo: Math 20C - Winter 2017
Lecture Notes

Date: February 13, 2017

Section:

- [ ] § 2.6 (cont.)

Topics Covered:

- [ ] More geometry of the gradient
- [ ] Equations of tangent planes to surfaces

From last time: If $f: \mathbb{R}^n \to \mathbb{R}$ (usually $n=2,3$) and if $\vec{u}$ is a unit vector, the directional derivative of f at $\vec{x}_0$ in the direction of $\vec{u}$ is:

$$\boxed{\begin{aligned}(D_{\vec{u}} f)(\vec{x}) &= \lim_{t \to 0} \frac{f(\vec{x}+t\vec{u}) - f(\vec{x})}{t} \\ &= (\nabla f)(\vec{x}) \cdot \vec{u}.\end{aligned}}$$

$(D_{\vec{u}} f)(\vec{x})$ tells us the rate of change in f in the direction of $\vec{u}$. Using the identity

$$(\nabla f)(\vec{x}) \cdot \vec{u} = \|\nabla f(\vec{x})\| \|\vec{u}\| \cos\theta$$ where θ is the angle between $\nabla f(\vec{x})$ and $\vec{u}$, we saw:

1. $(\nabla f)(\vec{x})$ points in the direction of steepest increase,
2. The maximal rate of increase of f at $\vec{x}$ is $\|\nabla f(\vec{x})\|$.

Example: Farmer Corey is standing near the middle of his corn field in the great city of Omaha, NE (say he is standing at the point $(0,1)$). The temperature at the point (x,y) is given by $T(x,y) = 30e^{-(x-1)^2-(2y-1)^2}$. very cold! What direction should he walk to warm up the fastest?

Farmer Corey is

Sol: He should walk in the direction of $(\nabla T)(0,1)$.

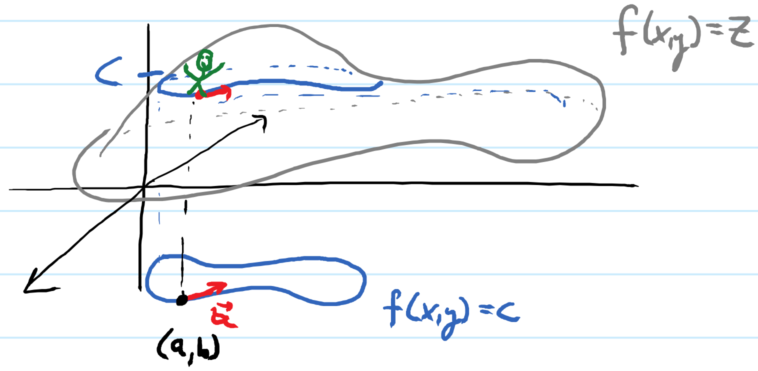
$$\nabla T = \left(-60(x-1)e^{-(x-1)^2-(2y-1)^2},\ -120(2y-1)e^{-(x-1)^2-(2y-1)^2}\right)$$

$$(\nabla T)(0,1) = \left(60e^{-2},\ -120e^{-2}\right).$$

Level sets revisited: If $f:\mathbb{R}^n \to \mathbb{R}$, and c is a constant, the level set of height c is the set of points $(x_1, \ldots, x_n)$ that satisfy $f(x_1, \ldots, x_n) = c$.

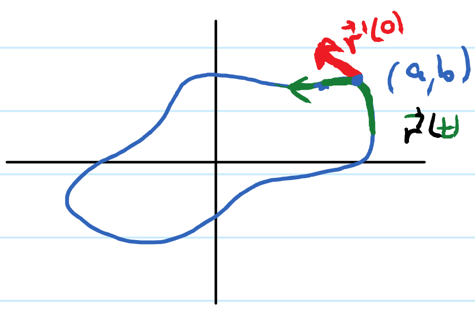
Suppose now that $f:\mathbb{R}^2 \to \mathbb{R}$, and that $\vec{u}$ is a unit vector that is tangent to the level curve $f(x,y) = c$.

If you walk on the graph of f along the level curve $f(x,y) = c$, you will not be ascending nor descending since your height will always be c.



Therefore, we should expect $(D_{\vec{u}} f)(a,b) = 0$.

Let's prove this: Let $\vec{r}(t)$ be a parametrization of the level curve near (a,b), such that $\vec{r}(0) = (a,b)$ and $\vec{r}\,'(0) = \vec{u}$.



Since f is constant on the level curve, and since $\vec{r}(t)$ is a point on the level curve for every t, we have $f(\vec{r}(t)) = c$. But then

$$\frac{d}{dt}(f \circ \vec{r})(0) = \frac{d}{dt}(c)$$

$$\Rightarrow \quad \nabla f(\vec{r}(0)) \cdot \vec{r}\,'(0) = 0$$

$$\Rightarrow \quad \nabla f(a,b) \cdot \vec{u} = 0$$

but then $(D_{\vec{u}} f)(a,b) = \nabla f(a,b) \cdot \vec{u} = 0$, which is what we wanted.

Remark: As a corollary to the above argument, we see that if $\vec{u}$ is a unit vector that is tangent to the level curve at (a,b), then

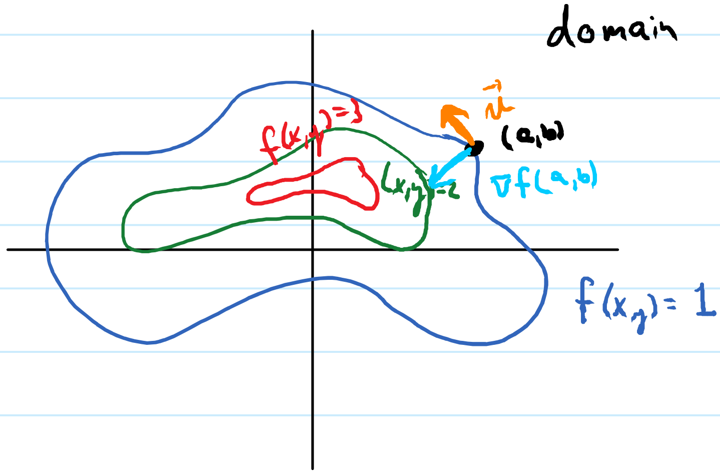
$$\nabla f(a,b) \cdot \vec{u} = 0$$
$$\iff \nabla f(a,b) \perp \vec{u}$$

This tells us that:

<u>The gradient vector is orthogonal to level sets!</u>

Ex:

domain of f.

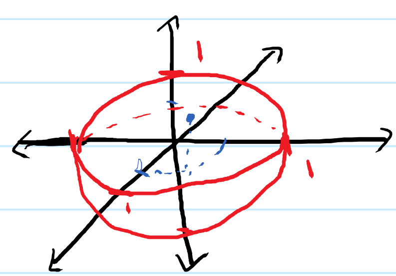


Rate of change of f is 0 in the $\vec{u}$ direction and is maximal in the direction of $\nabla f(a,b)$.

Application: Tangent planes to surfaces:

<u>Problem:</u> Find the equation of the plane that is tangent to the sphere $x^2+y^2+z^2=1$ at the point $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

picture:



Idea: We could try to solve for z in terms of (x,y). This way we write z as a fcn of x and y, and we can use our old formula for tangent planes. However, $x^2+y^2+z^2=1$

$$\Rightarrow \quad z = \pm\sqrt{1-x^2-y^2}.$$

The $\pm$ tells us z is not a fcn of x and y, so we try something different:

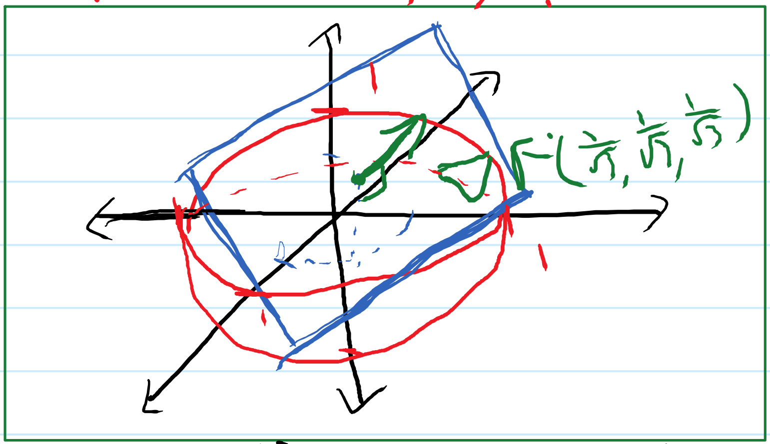
Let $F(x,y,z) = x^2+y^2+z^2$. Then the sphere $x^2+y^2+z^2=1$, is just the level surface of F with height 1:

$$F(x,y,z)=1 \iff x^2+y^2+z^2=1.$$

Since gradients are orthogonal to level sets,

$(\nabla F)\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is orthogonal to the sphere $x^2+y^2+z^2=1$.

In particular, $(\nabla F)\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is normal to the tangent plane at $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.



$\nabla F = (2x, 2y, 2z)$.

$$\boxed{\nabla F\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)}$$

Now we have a normal vector:

$\vec{N} = \left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$, and a point on the plane $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

So we can write the equation of the plane:

$$\frac{2}{\sqrt{3}}x + \frac{2}{\sqrt{3}}y + \frac{2}{\sqrt{3}}z = \left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)\cdot\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\frac{2}{\sqrt{3}}x + \frac{2}{\sqrt{3}}y + \frac{2}{\sqrt{3}}z = \frac{2}{3} + \frac{2}{3} + \frac{2}{3}$$

$$\boxed{\frac{2}{\sqrt{3}}x + \frac{2}{\sqrt{3}}y + \frac{2}{\sqrt{3}}z = 2}$$

Note: To make this look nicer, we could multiply by $\frac{\sqrt{3}}{2}$ to get:

$$\boxed{x + y + z = \sqrt{3}}$$

Example: Find an equation to the tangent plane to the surface

$$e^{y^2}z^4x - x^4z = 0 \quad \text{at} \quad (1, 0, 1).$$

Note: The set of points that satisfy this equation forms some surface in $\mathbb{R}^3$. I am not able to draw it, and I certainly can't solve for z to get a fcn of x and y. Let's use the same strategy as last time:

Let $F(x,y,z) = e^{y^2}z^4x - x^4z$. Then the surface $e^{y^2}z^4x - x^4z = 0$ is the level surface $F(x,y,z) = 0$.

Therefore, $\nabla F(1,0,1)$ is $\perp$ to the surface at $(1,0,1)$ and hence $\nabla F(1,0,1)$ is normal to the tangent plane.

$$\nabla F = \left(e^{y^2}z^4 - 4x^3z,\ 2ye^{y^2}z^4x,\ 4e^{y^2}z^3x - x^4\right)$$

$$\boxed{\nabla F(1,0,1) = (-3, 0, 3)}$$

Now we have our normal vector: $\vec{N} = (-3, 0, 3)$, and a point on the plane: $(1, 0, 1)$.

So we get the plane:

$$-3x + 0 \cdot y + 3z = (-3, 0, 3) \cdot (1, 0, 1)$$

$$\Rightarrow \quad \boxed{-3x + 3z = 0}$$