Longo: Math 20C - Winter 2017
Lecture Notes

Date: February 15, 2017

Section:
- [ ] Equations of tangent planes via implicit differentiation (not in the book)
- [ ] § 3.1

Topics Covered:
- [ ] Implicit differentiation
- [ ] Iterated partial derivatives and Clairaut's Theorem on the equality of mixed partials

Last time: Given a surface defined by some equation $F(x,y,z)=c$ (for example, $x^2+y^2+z^2=3$) we saw that the vector $\nabla F(a,b,c)$ is orthogonal to the surface $F(x,y,z)=c$ (because gradients are $\perp$ to level sets!)

This gave us a way to calculate eqns of tangent planes to arbitrary surfaces.

Another option (that is not mentioned in the book as far as I know) is to use implicit differentiation.

<u>Example from before:</u> Find the eqn of the plane tangent to the sphere $x^2+y^2+z^2=3$ at $(1,1,1)$ using implicit differentiation.

<u>Idea:</u> We saw last time that we cannot solve for z, and express z as a fcn of x,y. In this case, we say z is not an explicit fcn of x,y. Nevertheless, since $x^2+y^2+z^2=3$, there is a clear restriction for what z can be depending on x and y. So we say z is implicitly a fcn of x and y. It therefore makes sense to compute $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$. We can then use the "old" tan. plane formula:

$$(*) \quad z = c + \frac{\partial z}{\partial x}(a,b)(x-a) + \frac{\partial z}{\partial y}(a,b)(y-b)$$

where (a,b,c) is the point of tangency.

$\boxed{\text{Q?}}$ How do we find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$?

$\boxed{\text{Ans:}}$ Use implicit differentiation on the equation $x^2+y^2+z^2=3$.

Take the partial w.r.t x: (differentiate the eqn w.r.t. x: we still pretend y is constant, but when we see z, we diff. w.r.t. z, then multiply by $\frac{\partial z}{\partial x}$. This is just like implicit diff. in calc. 1, except y is const. Note, this process works because of the chain rule).

$$\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)=\frac{\partial}{\partial x}(3)$$

$$2x+\underbrace{2z\cdot\frac{\partial z}{\partial x}}_{\text{chain rule}}=0$$

$$\Rightarrow 2z\frac{\partial z}{\partial x}=-2x$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x}=\frac{-x}{z}}$$

Take partial w.r.t y: (This time x is const.)

$$\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)=\frac{\partial}{\partial y}(3)$$

$$\Rightarrow 2y+2z\cdot\frac{\partial z}{\partial y}=0$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial y}=\frac{-y}{z}}$$

Since the point we are looking at is $P_0=(1,1,1)$, we have

$$\boxed{\begin{aligned}\left(\frac{\partial z}{\partial x}\right)&=\frac{-1}{1}=-1\\ \frac{\partial z}{\partial y}&=\frac{-1}{1}=-1\end{aligned}}$$

Now using eqn $(*)$, the tan. plane is given by

$$z=1-1(x-1)-1(y-1)$$

$$\Rightarrow \boxed{z+x+y=3} \quad \text{which is the same as before.}$$

Remark: ① This process doesn't always work: sometimes you are forced to have 0 in the denominator.
② We are really doing the same thing as before but wording it differently.

§3.1: Iterated partial derivatives:

The next major goal in this class will be to Optimize fcns of two variables. Meaning we want to:

① Find local/global max/mins.
② Find critical points (points where the tan plane is horizontal)
③ Classify crit. pts. as local max / local min / other by using Concavity. In calc. I, this is called the Second derivative test.

We start by addressing goal ③ and discussing the multivariable version of second derivatives:

Iterated Partial Derivatives:

Start with a fcn $f(x,y)$. Then we can take partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$. Now

$\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are themselves functions of x and y, so it makes sense to take partial derivatives of $\frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial y}$ w.r.t. x,y. The result will be called a Second order partial derivative.

Notation: ① $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$ (or f_{xx}) means take the x partial twice.

② $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$ (or f_{xy}) means take x-partial then the y-partial.

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④ $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$ (or f_{yx}) means take y-partial, then the x-partial

We will talk about what these represent soon but first:

<u>Examples:</u> Find <u>ALL</u> second order partials for:

① $f(x,y) = e^{-y}\ln(x) - x^2y^2$.

$$\frac{\partial f}{\partial x} = \frac{e^{-y}}{x} - 2xy^2$$

$$\frac{\partial f}{\partial y} = -e^{-y}\ln(x) - 2x^2y$$

$$\boxed{\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{-e^{-y}}{x^2} - 2y^2 \\ \frac{\partial^2 f}{\partial y \partial x} &= \frac{-e^{-y}}{x} - 4xy \\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{-e^{-y}}{x} - 4xy \\ \frac{\partial^2 f}{\partial y^2} &= e^{-y}\ln(x) - 2x^2 \end{aligned}}$$

② $g(x,y) = y^3x - x^2y^2 + 4x^2y$

$$\frac{\partial g}{\partial x} = y^3 - 2xy^2 + 8xy$$

$$\frac{\partial^2 g}{\partial x^2} = 2y^2 + 8y$$

$$\frac{\partial^2 g}{\partial y \partial x} = 3y^2 - 4xy + 8x$$

$$\frac{\partial g}{\partial y} = 3y^2x - 2x^2y + 4x^2$$

$$\frac{\partial^2 g}{\partial x \partial y} = 3y^2 - 4xy + 8x$$

$$\frac{\partial^2 g}{\partial y^2} = 6yx - 2x^2$$

③ $h(x,y) = \cos(xy) + y^2$

$\frac{\partial h}{\partial x} = -\sin(xy)\cdot y$

$\frac{\partial h}{\partial y} = -x\sin(xy) + 2y$

$\frac{\partial^2 h}{\partial x^2} = -y^2\cos(xy)$

$\frac{\partial^2 h}{\partial y\partial x} = -\sin(xy) + y(-x\cos(xy))$

$= -\sin(xy) - xy\cos(xy)$

$\frac{\partial^2 h}{\partial x\partial y} = -\sin(xy) - x(\cos(xy)\cdot y)$

$= -\sin(xy) - xy\cos(xy)$

$\frac{\partial^2 h}{\partial y^2} = -x^2\cos(xy) + 2$

In each of these examples, we see that the mixed partials $\frac{\partial^2 f}{\partial x\partial y}$, $\frac{\partial^2 f}{\partial y\partial x}$ are equal!

It turns out that if the fcns are "nice enough", this will always be true.

Theorem: (Clairaut): If the mixed partials of $f(x,y)$ and are continuous, then they are equal:

$$\frac{\partial^2 f}{\partial xy} = \frac{\partial^2 f}{\partial yx}$$

Rmk: This is a great way to check your work!

Example: Does there exist a fcn with continuous 2^{nd} order partials such that:

$\frac{\partial f}{\partial x} = 4x^2 + y$ $\qquad$ $\frac{\partial f}{\partial x} = 4y^3x + x.$

Sol: If there were, then

$$\frac{\partial^2 f}{\partial y \partial x} = 1 \qquad \text{while} \qquad \frac{\partial^2 f}{\partial x \partial y} = 4y^3 + 1$$

By Clairaut's thm, this is impossible.