Longo: Math 20C - Winter 2017
Lecture Notes

Date: February 17, 2017

Section:

- [ ] §3.3

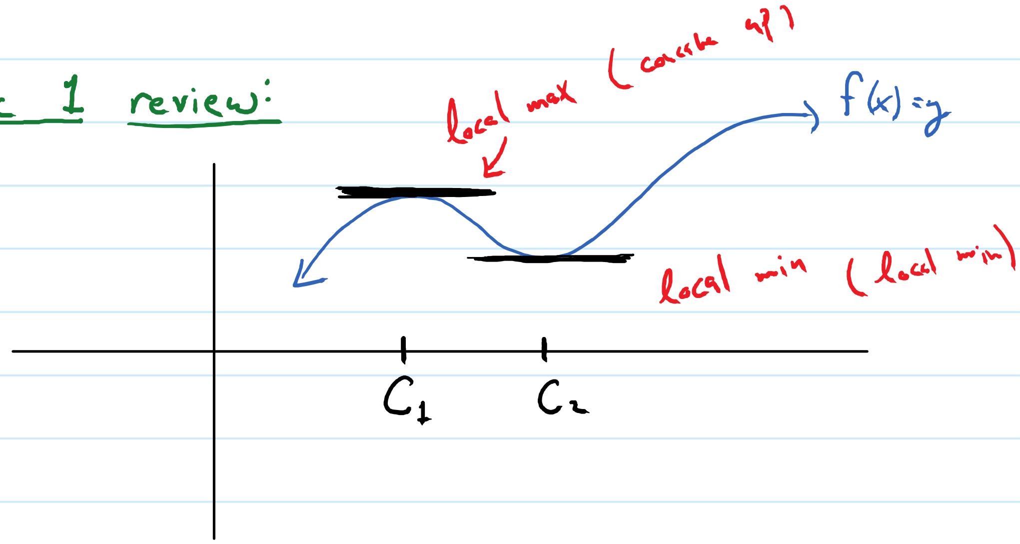
Topics Covered:

- [ ] Review of local maxima and minima for functions of one variable
- [ ] Local maxima, local minima, saddle points for functions of two variables

§3.2 Part 1: Optimization - Local extrema and saddle points:

Def: A fcn f (of 1, 2, or 3 input variables for us) has a local maximum at x_0 if for all other x near x_0, $f(x_0) \geq f(x)$.
The notion of local minimum is defined similarly.

Quick Calc 1 review:



A cont. fcn, f, potentially has local extrema at critical points:

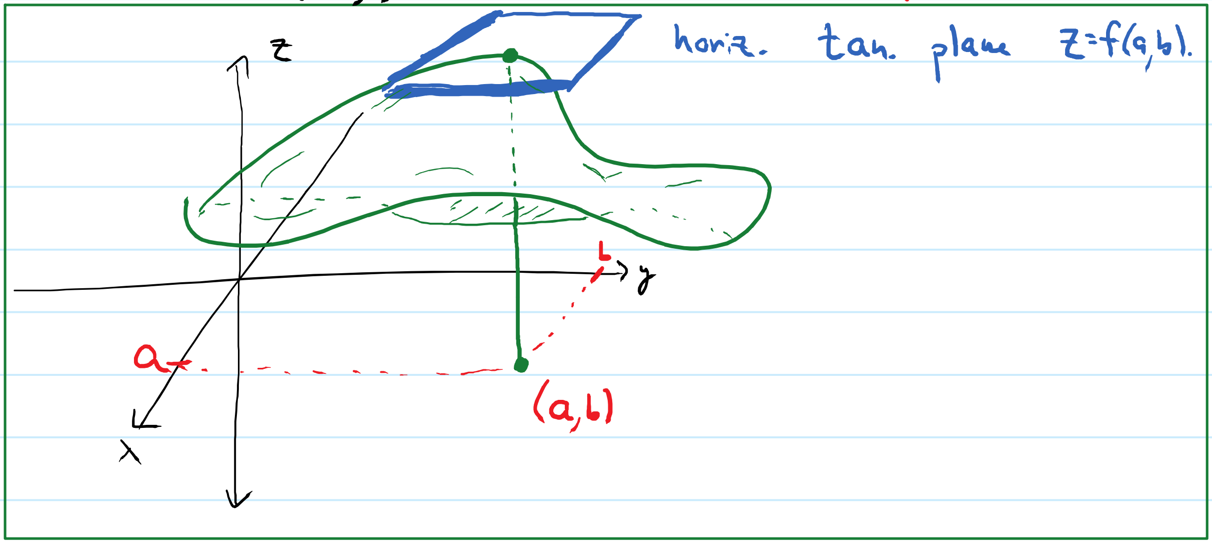
I.e., points where $f'(x)=0$ (horizontal tangent line) or where f' is undefined.

We can test if a critical point, a, gives a local max (resp. local min) by checking if $f''(a)>0$, which means the graph is concave down at a (resp. $f''(a)<0$, which means the graph is concave up at a).

Local extrema for fcns $z=f(x,y)$: A fcn $f(x,y)=z$ potentially has a local max. or local min at points (a,b) where the tangent plane: $z=f(a,b)+\frac{\partial f}{\partial x}(a,b)(x-a)+\frac{\partial f}{\partial y}(a,b)(y-b)$ is horizontal. It is not hard to see that this happens

(or is undefined)

iff both $\frac{\partial f}{\partial x}(a,b)=0$ and $\frac{\partial f}{\partial y}(a,b)=0$,
i.e., when $\nabla f(a,b)=0$. (or when either partial is undefined.



Example: ① Find all crit. pts of $f(x,y)=x^2-3xy+5x-2y+6y^2+8$.

Sol: $\nabla f=(2x-3y+5,\ -3x-2+12y)$
Both partials are cont. everywhere so we must solve the system of equations:

$$\left.\begin{array}{ll}\frac{\partial f}{\partial x}: & 2x-3y+5=0\\ \frac{\partial f}{\partial y}: & -3x-2+12y=0\end{array}\right\} \Rightarrow \left.\begin{array}{l}2x-3y=-5\\ -3x+12y=2\end{array}\right\} \Rightarrow \boxed{3y=2x+5}$$

plug into eqn 2:

$$-3x+4(3y)=2$$
$$-3x+4(2x+5)=2$$
$$\Rightarrow 5x+20=2$$
$$\Rightarrow \boxed{x=\frac{-18}{5}} \Rightarrow y=\tfrac{1}{3}\left(2\left(\tfrac{-18}{5}\right)+5\right)$$
$$y=\tfrac{1}{3}\left(\tfrac{-11}{5}\right)$$
$$\boxed{y=\frac{-11}{15}}$$

so there is one crit. pt. at $\boxed{\left(\frac{-18}{5}, \frac{-11}{15}\right)}$

② $g(x,y)=8y^3x+12x^2-24xy$.

p

Sol: g is a polynomial and is therefore diff'ble everywhere

So we try to solve: $\nabla g = 0$

$\nabla g = (8y^2+24x-24y,\ 16yx-24x) = 0$. So we solve the system:

$$\boxed{\begin{array}{ll} \text{I} & 8y^2+24x-24y=0 \\ \text{II} & 16yx-24x \stackrel{?}{=} 0 \end{array}}$$

II $\Rightarrow$ $8x(2y-3)=0$ $\Rightarrow$ either $\boxed{x=0}$ or $\boxed{y=\frac{3}{2}}$
(possibly both).

Case 1: If $x=0$, I $\Rightarrow$ $8y^2-24y=0$
$\Rightarrow 8y(y-3)=0$
$\Rightarrow \boxed{y=0}$ or $\boxed{y=3}$

So from case 1: we get two crit. pts $\boxed{(0,0)}$, $\boxed{(0,3)}$

Case 2: If $y=\frac{3}{2}$, I $\Rightarrow$ $8\left(\frac{3}{2}\right)^2+24x-24\left(\frac{3}{2}\right)=0$
$\Rightarrow 8\left(\frac{9}{4}\right)+24x-36=0$
$\Rightarrow 18+24x-36=0$
$\Rightarrow 24x=18$
$\Rightarrow \boxed{x=\frac{3}{4}}$

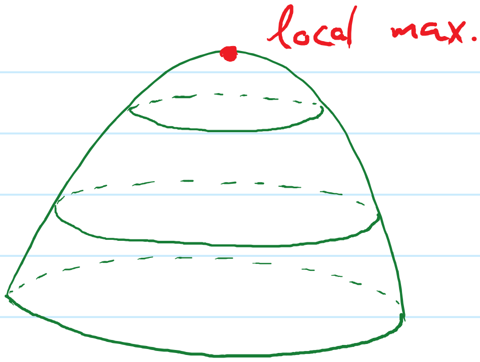
So case 2 gives us a crit. pt. $\boxed{\left(\frac{3}{4},\frac{3}{4}\right)}$

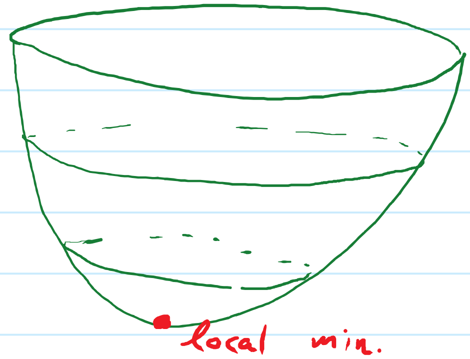
So g has 3 crit. pts: $\boxed{(0,0),(0,3),\left(\frac{3}{4},\frac{3}{4}\right)}$.

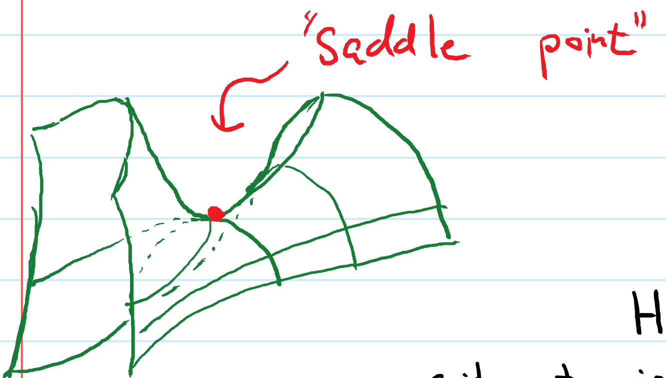
Classification of critical points:

(has cont. 2nd partials)

For "nice" fcns $f(x,y)$ of 2 variables, there are three types of critical points:







If we zoom in on the graph of f at a crit. pt., it looks similar to one of these.

How do we know if a crit. pt. is a local max./min./saddle?.

Thm: (Second derivative test): Let $f(x,y)$ be a fcn with cont. 2^{nd} order partials. Let (a,b) be a crit. pt. of f. Let

$$D(a,b) = \left(\left(\frac{\partial^2 f}{\partial x^2}\right)(a,b)\right)\left(\frac{\partial^2 f}{\partial y^2}(a,b)\right) - \left(\left(\frac{\partial^2 f}{\partial x \partial y}\right)(a,b)\right)^2.$$

D is called the discriminant at (a,b). Then

(I) If $D(a,b) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a,b) > 0$, f has a local min at (a,b).

(II) If $D(a,b) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a,b) < 0$, f has a local max at (a,b).

(III) If $D(a,b) < 0$, f has a saddle point at (a,b).

Rmk: ① If $D(a,b) = 0$, the test does not give us any information!

② $D(x,y)$ is the determinant of the Hessian matrix of f: $H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$

③ A proper explanation of the test would require a discussion of multivariable Taylor Polynomials, and a development of some linear algebra that is beyond the scope of math 18, so we will skip it. See the book for details.

Example: Classify the critical points: $(0,0)$, $(0,3)$, $(\frac{45}{48}, \frac{3}{4})$ for the fcn $g(x,y) = 8y^2x + 12x^2 - 24xy$.

Sol: Recall $\frac{\partial f}{\partial x} = 8y^2 + 24x - 24y$ $\qquad \frac{\partial f}{\partial y} = 16yx - 24x$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial x^2} = 24 \qquad \frac{\partial^2 f}{\partial y^2} = 16x$$

$$\frac{\partial^2 f}{\partial y \partial x} = 16y - 24 \quad = \quad \frac{\partial^2 f}{\partial x \partial y} = 16y - 24$$

Therefore, $D(x,y) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial y \partial x}\right)^2$
$= 24(16x) - (16y - 24)^2.$

For $(0,0)$: $D(0,0) = 24(16(0)) - (-24)^2 < 0$

Therefore $(0,0)$ is a saddle point.

For $(0,3)$: $D(0,3) = 24(16(0)) - (16(3) - 24)^2$
$= 0 - (16(3) - 24)^2 < 0$

Therefore $(0,3)$ is a saddle point.

For $(\frac{3}{4}, \frac{3}{4})$:

$$D\left(\frac{3}{4},\frac{3}{4}\right) = (24)(16)\left(\frac{3}{4}\right) - \left(16\left(\frac{3}{4}\right) - 24\right)^2$$
$$= (24)(12) - (12-24)^2$$
$$= 2(12^2) - (-12)^2 > 0.$$

$\left(\frac{3}{4}, \frac{3}{4}\right)$ may be a local max. or min. So let's check

$\frac{\partial^2 f}{\partial x^2}\left(\frac{3}{4},\frac{3}{4}\right) = 24 > 0$. Therefore, f has a local min. at $\left(\frac{3}{4}, \frac{3}{4}\right)$.