Longo: Math 20C - Winter 2017
Lecture Notes

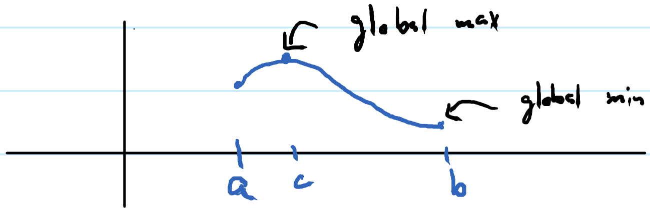
Date: February 22, 2017

Section:
- [ ] § 3.3 (cont.)

Topics Covered:
- [ ] Closed sets
- [ ] Extreme Value Theorem
- [ ] Global Extrema

§3.3 (cont.):

Recall: If $f(x)$ is a continuous fcn on a closed interval $[a,b]$, then f has a global max. and global min in the interval $[a,b]$. Moreover, the global extrema occur either at critical pts. or the endpts.

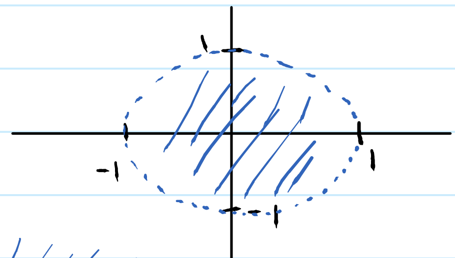


If the domain of a fcn is a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$ what should we mean by "endpoints".

Informal definitions: ① A subset, $\mathcal{U}$, of $\mathbb{R}^n$ is called **bounded** if $\mathcal{U}$ "does not go off to ∞".

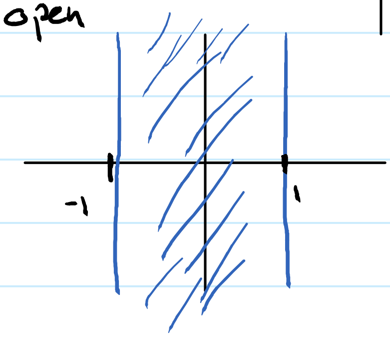
② A subset, $\mathcal{U}$, of $\mathbb{R}^n$ is called **closed** if $\mathcal{U}$ contains its boundary.

Examples: ① $\{(x,y) \mid x^2+y^2<1\}$



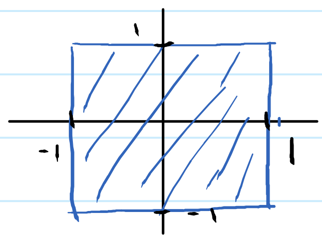
bounded, but not open

② $\{(x,y) \mid -1 \le x \le 1\}$



Includes its boundary, but goes off to ∞. This set is closed, but not bounded.

③ $\{(x,y) \mid -1 \le x \le 1,\ -1 \le y \le 1\}$



This set is closed & bounded.

Rmk: A subset, $\mathcal{U}$, of $\mathbb{R}^n$ that is closed <u>and</u> bounded is called compact.

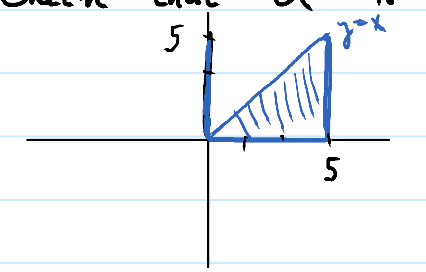
Thm: (Extreme Value Theorem): If f is a cont. fcn defined on a closed and bounded subset $\mathcal{U}$ of $\mathbb{R}^n$ (for us $n=2$ or 3) then f has a global max and a global min. on $\mathcal{U}$. Moreover, the global extrema must occur at critical points, or on the boundary, $\partial\mathcal{U}$, of $\mathcal{U}$.

Examples: ① Find global extrema for the fcn

$$f(x,y) = x^2y - x - 3xy^2$$

on the square $\mathcal{U} = \{(x,y) \mid 0 \le x \le 5,\ 0 \le y \le x\}$

Sol: Check that $\mathcal{U}$ is closed & bounded.



yes, $\mathcal{U}$ is closed and bounded.

Step 1: Find critical points on the interior.

Since f is a polynomial, it is diff'ble everywhere, so we must solve $\nabla f = \vec{0}$.

$$\nabla f = (2xy - 1 - 3y^2,\ x^2 - 6xy) = \vec{0}$$

$$\Leftrightarrow \quad \text{Ⓘ } 2xy - 1 - 3y^2 = 0$$

$$\text{Ⓘⓘ } x^2 - 6xy = 0$$

Ⓘⓘ $\Rightarrow$ $x(x-6y) = 0 \Rightarrow$ either $x=0$ or $x=6y$

Since the only point where $x=0$ is on the boundary, (which we will check separately) we can ignore the case where $x=0$.

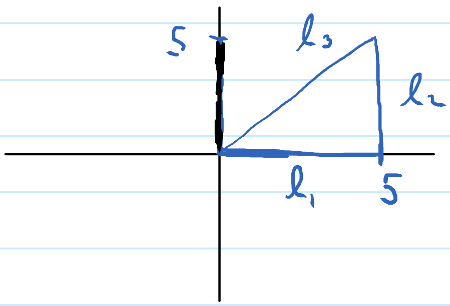
So we get $\boxed{x=6y}$. Plug this back into eqn Ⓘ

$$2(6y)y - 1 - 3y^2 = 12y^2 - 1 - 3y^2 = 0 \Rightarrow 9y^2 = 1 \Rightarrow$$

$\boxed{y = \pm\frac{1}{3}}$, Since every point in $\mathcal{U}$ has a positive y-value, $y = -\frac{1}{3}$ is not in the domain, so we can disgard it, and we get $\boxed{y = \frac{1}{3}}$. Since $x = 6y$, $\boxed{x = 2}$

So we have one critical pt. on the interior of $\mathcal{U}$: $\boxed{(2, \frac{1}{3})}$

Step 2: Find potential max/mins along the boundary:



Note that the boundary can be split into three line segments that are easy to parametrize. We must look for potential extrema on all three sides.

l_1: l_1 can be described as: $y = 0$, and $0 \leq x \, 5$.
So the fcn f restricted to l_1 becomes
$$f_1(x) = f(x, 0) = -x \quad , \quad 0 \leq x \leq 5.$$

This is a fcn of one variable defined on a closed interval. it's max./mins. happen at crit. pts or the endpts.

$f_1'(x) = -1 \rightarrow$ no crit. pts., so we only need the endpts. $x = 0$, $x = 5$. Since $y = 0$ on l_1, we pick up two more points that could potentially be global extrema: $\boxed{(0,0)}$, $\boxed{(5,0)}$

$\underline{\ell_2}$: ℓ_2 can be described by: $x=5$, $0 \le y \le 5$. So f restricted to ℓ_2 becomes

$$f_2(y) = f(5,y) = 25y - 5 - 15y^2, \qquad 0 \le y \le 5.$$

Again, this is a one variable fcn, so we use calc 1 techniques.

Find crit. pts. for f_2: $f_2'(y) = -30 + 25y = 0 \Rightarrow \boxed{y = \frac{6}{5}}$

We must also consider the end pts. $y=0$, $y=5$.

Since $x=5$ on ℓ_2, we get two new points that could potentially be max./mins. for f: $\boxed{(5, \frac{6}{5}), (5,5).}$

Lastly,

$\underline{\ell_3}$: ℓ_3 can be described as: $y=x$, $0 \le x \le 5$. So f restricted to ℓ_3 becomes:

$$f_3(x) = f(x,x) = x^3 - x - 3x^3$$
$$= -2x^3 - x \qquad 0 \le x \le 5$$

which is a fcn of one variable.

Find crit. pts. for f_3: $f_3'(x) = -6x^2 - 1 = 0$

$\Rightarrow x^2 = \frac{-1}{6}$ which is impossible.

So we only need to look at the endpts:

$x=0$, $x=5$.

On ℓ_3, $y=x$ so our potential points on ℓ_3 are $(0,0)$, $(5,5)$, which we already had.

$\underline{\text{Step 3:}}$ Plug in all of the potential points to see which gives us a global max, and which gives a global min.

Our potential points are: $(2, \frac{1}{3})$, $(0,0)$, $(5,0)$, $(5,5)$, $(5, \frac{6}{5})$.

plug each into $f(x,y) = x^2y - x - 3xy^2$:

$f(0,0) = 0$

$f(5,0) = -5$ Smallest

$f(5,5) = 5^2 \cdot 5 - 5 - 3 \cdot 5^2 \cdot 5 = -255$

$f(2, \frac{1}{3}) = \frac{4}{3} - 2 - \frac{6}{9} = \frac{4}{3} - \frac{6}{3} - \frac{2}{3} = -\frac{4}{3}$

$f(5, \frac{6}{5}) = 25\left(\frac{6}{5}\right) - 5 - 3(5)\left(\frac{36}{25}\right)$

$= 30 - 5 - \frac{108}{5}$

$= \frac{17}{5}$ largest.

So f has a global max. of $\frac{17}{5}$ at $(5, \frac{6}{5})$ and f has a global min. of -255 at $(5,5)$.

Summary: Step 1: Find critical points on the interior of U.

Step 2: (A) Find a way to describe the boundary so that f restricted to the boundary becomes a fcn of 1-variable.

(B) Find all potential points on the boundary using Calc 1.

Step 3: Plug all of the points you found in step 1, 2 into f to find the global m