Longo: Math 20C - Winter 2017
Lecture Notes

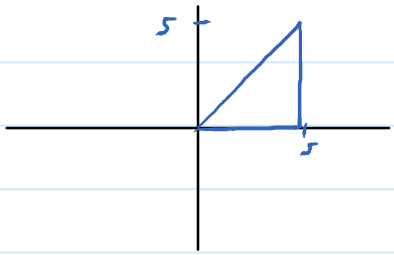
Date: February 24, 2017

Section:
- §3.4

Topics Covered:
- Optimization using the method of Lagrange Multipliers

§3.4: The method of Lagrange multipliers

Last time, we saw how to find extreme values of a cont. fcn f, constrained to the triangle



We did this by finding a nice parametrization of the sides of the triangle and then composing with f. Then we turned the problem into a calc 1 problem. If the constraint is hard to parametrize, then this techique isn't ideal.

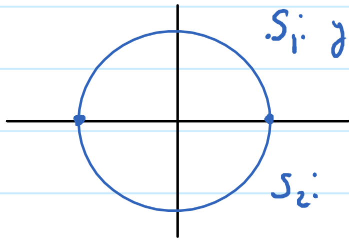
Example: Say we want to find the max./min. of the fcn $f(x,y) = xy$ subject to the constraint $x^2+y^2=1$.

I.e., we want to find max./min. of f restricted to the domain $C = \{(x,y) \mid x^2+y^2=1\}$.

Let's first note that f is continuous and the domain is the unit circle, which is closed and bounded. So by the extreme value thm, f does have global max./mins.

Second, note that the graph of f is the saddle, so we should already expect max at $(\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}})$, $(\frac{-2}{\sqrt{2}}, \frac{-2}{\sqrt{2}})$ and min at $(\frac{2}{\sqrt{2}}, -\frac{2}{\sqrt{2}})$, $(\frac{-2}{\sqrt{2}}, \frac{2}{\sqrt{2}})$.

Using techiques from last time, we can split the circle into an upper hemisphere and lower hemisphere, each of which can be parametrized:



S_1: $y = \sqrt{1-x^2}$, $-1 \le x \le 1$

S_2: $y = -\sqrt{1-x^2}$, $-1 \le x \le 1$

$\underline{S_1}$: f restricted to S_1 becomes

$$f_1(x) = f(x, \sqrt{1-x^2}) = x\sqrt{1-x^2} \quad . \quad -1 \le x \le 1.$$

Find critical pts: $f_1'(x) = \sqrt{1-x^2} + x\left(\frac{-x}{\sqrt{1-x^2}}\right) = 0$

$$\Rightarrow \quad 1 - x^2 - x^2 = 0$$
$$\Rightarrow \quad 2x^2 = 1$$
$$\Rightarrow \quad x^2 = \tfrac{1}{2}$$
$$\Rightarrow \quad x = \pm\tfrac{1}{\sqrt{2}} = \pm\tfrac{\sqrt{2}}{2}$$

when $x = \frac{\sqrt{2}}{2}$, $y = \sqrt{1-(\frac{\sqrt{2}}{2})^2} = \frac{\sqrt{2}}{2}$

$x = \frac{-\sqrt{2}}{2}$, $y = \sqrt{1-(\frac{-\sqrt{2}}{2})^2} = \frac{\sqrt{2}}{2}$

So two critical pts: $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

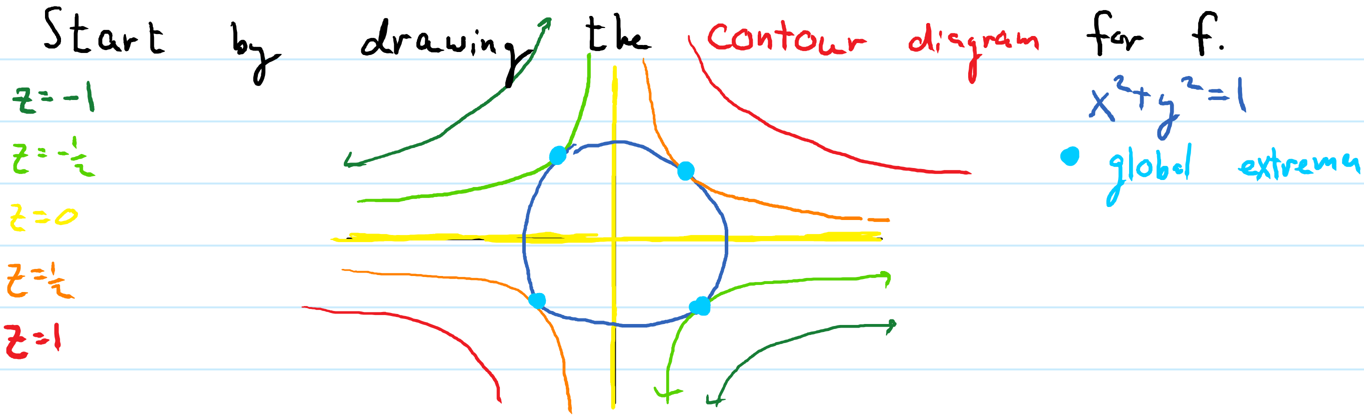
On S_2, we get critical pts at: $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$, $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$.

We must also include the end pts: $(1,0)$, $(-1,0)$.

plugging back into $f(x,y) = xy$, we see f has a global max of $\frac{1}{2}$ occuring at $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ and a global min. of $-\frac{1}{2}$ occuring at $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ and $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

Let's think of a better way to do this:

Start by drawing the contour diagram for f.

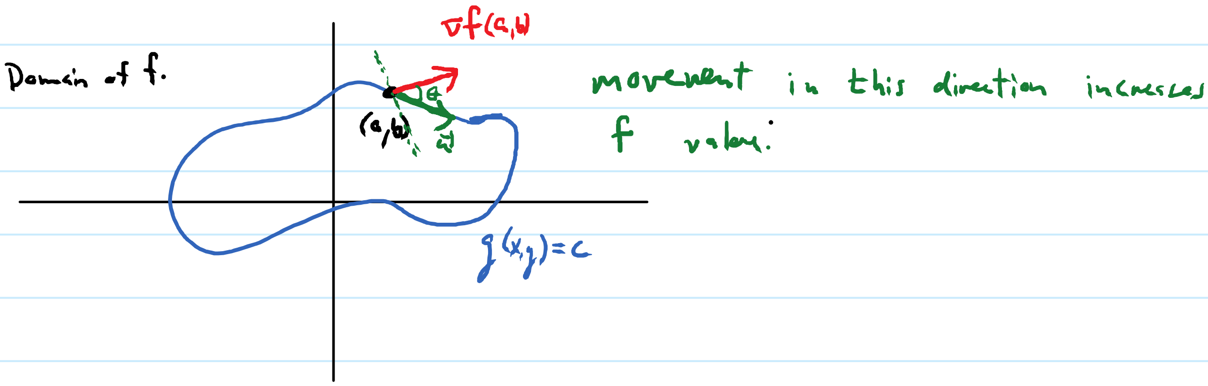


If we draw the constraint: $x^2+y^2=1$ on top of the contour diagram, we see that at the points where f achieves its global max./min., the constraint $x^2+y^2=1$ is tangent to the level curve of f.

In fact, this is how we find global extrema.

Why does this work? In the general setting, suppose f is cont., g is cont., and c is a constant. Suppose we want to find the max./min.s of $f(x,y)$ subject to the constraint $g(x,y)=c$. If (a,b) is a max. of f on $g(x,y)=c$, then we claim $\nabla f(a,b)$ be $\perp$ to the constraint curve.

<u>Proof in the special case where g, f are both diff'ble:</u> Assume $\nabla f(a,b)$ is not $\perp$ to $g(x,y)=c$. Then the angle θ between $\nabla f(a,b)$ and $g(x,y)=c$ is $<\frac{\pi}{2}$. Let $\vec{u}$ be a unit vector tangent to $g(x,y)=c$.



Then
$$\begin{aligned}[D_{\vec{u}} f](a,b) &= (\nabla f)(a,b)\cdot\vec{u}\\ &= \|\nabla f(a,b)\|\,\|\vec{u}\|\cos\theta\\ &= \|\nabla f(a,b)\|\cos\theta.\end{aligned}$$

Since $\theta<\frac{\pi}{2}$, $\cos\theta>0 \Rightarrow [D_{\vec{u}} f](a,b)>0$. This means if we walk along the constraint $g(x,y)=c$ in the $\vec{u}$ direction, the value of f increases. This contradicts the fact that (a,b)

was supposed to be a max of f on $g(x,y)=c$. ▣.

Corollary: ① Since $\nabla f(a,b)$ is $\perp$ to the level curve of f at (a,b) and $\nabla f(a,b)$ is $\perp$ to $g(x,y)=c$, the level curve of f at (a,b) is $\parallel$ (and hence tangent) to $g(x,y)=c$.

② Since $(\nabla g)(a,b)$ and $(\nabla f)(a,b)$ are both $\perp$ to $g(x,y)=c$, $\boxed{\nabla g(a,b) \text{ is } \parallel \text{ to } (\nabla f)(a,b).}$

Part ② of the corollary gives us:

Theorem: (The method of Lagrange multipliers): Let f, g be "nice" fcns from $\mathbb{R}^n \to \mathbb{R}$ (f, g have continuous partials). If $\vec{x}_0$ is a global min. or max. of f subject to the constraint $g(\vec{x})=c$, and if $(\nabla g)(\vec{x}_0) \neq \vec{0}$, then $(\nabla g)(\vec{x}_0) \parallel (\nabla f)(\vec{x}_0)$. I.e., there is a constant, λ, such that

$$(\nabla f)(\vec{x}_0) = \lambda (\nabla g)(\vec{x}_0)$$

Warning: ① This theorem does not guarantee the existence of extreme values. If $f|_{g=c}$ does have extreme values, this is how you find them.

② The points where $\nabla g = \vec{0}$ are considered critical points. This almost never comes up.

Example: Use the method of Lagrange multipliers to find extreme values of $f(x,y) = xy$ subject to the constraint $x^2+y^2=1$.

Sol: Since $x^2+y^2=1$ is closed and bounded, extreme values exist.

Let $g(x,y) = x^2+y^2$. Then the constraint is $g(x,y)=1$.

Note! $\nabla g(x,y) = (2x, 2y)$. Since $\nabla g = \vec{0}$ only at $(0,0)$, which is not on the unit circle $x^2+y^2=1$, we can move on to solving the Lagrange Condition:

$$\nabla f = \lambda \nabla g.$$

Since $\nabla f = (y,x)$, we must solve

$$(y,x) = \lambda(2x, 2y). \quad \& \quad x^2+y^2=1$$

$\Rightarrow$ I $\quad y = \lambda 2x$

II $\quad x = \lambda 2y$

III $\quad x^2+y^2=1$

General strategy 1: Solve for λ in terms of x,y.

I $\quad y = 2\lambda x \Rightarrow$ either $\lambda = \frac{y}{2x}$ or $x=0$, in which case, dividing by x doesn't make sense!

However, if $x=0$, we get $y = 2\lambda(0) = 0$, which implies $(x,y) = (0,0)$ which is not on the unit circle. Therefore it is impossible for $x=0$, and we therefore know $\boxed{\lambda = \frac{y}{2x}}$

II $\quad x = 2\lambda y \Rightarrow \lambda = \frac{x}{2y}$ or $y=0$, in which case dividing by 0 doesn't make sense. By the same reasoning as above, y cannot be 0. Therefore $\boxed{\lambda = \frac{x}{2y}}$

$$\Rightarrow \frac{y}{2x} = \frac{x}{2y} \Rightarrow 2y^2 = 2x^2 \Rightarrow \boxed{y^2 = x^2 \Rightarrow y = \pm x}$$

III $\quad x^2+y^2=1 \Rightarrow x^2+x^2=1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow \boxed{x = \frac{1}{\sqrt{2}} \text{ or } \frac{-1}{\sqrt{2}}.}$

If $x = \frac{1}{\sqrt{2}}$, $y = \pm x = \pm\frac{1}{\sqrt{2}}$. So we get two critical points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.

If $x = -\frac{1}{\sqrt{2}}$, $y = \pm x \Rightarrow y = \mp\frac{1}{\sqrt{2}}$, So we have two more critical pts: $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Finally!

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{2}$$
$$f\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = \frac{-1}{2}$$
$$f\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = \frac{1}{2}$$
$$f\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{-1}{2}$$

global max of $\frac{1}{2}$ at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
$\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

global min. of $\frac{-1}{2}$ at $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Next time we will do examples until we run out of time.