Longo: Math 20C - Winter 2017
Lecture Notes

Date: February 27, 2017

Section:

- [ ] §3.4

Topics Covered:

- [ ] Examples using the method of Lagrange multipliers

From last time:

Let f, g be "nice" fcns (f, g have continuous partials). Suppose $\vec{x}_0$ is a global extreme of f subject to the constraint $g = c$. If $(\nabla g)(\vec{x}_0) \neq \vec{0}$, then

$$\boxed{(\nabla f)(\vec{x}_0) = \lambda (\nabla g)(\vec{x}_0)}$$

Examples: ① Find the max and min of $f(x,y,z) = x - y + 2z$ on the closed and bounded domain
$\{(x,y,z) \mid x^2+y^2+z^2 = 2\}$

Note: the domain is the set of all point on the sphere of radius $\sqrt{2}$. In particular, the domain is closed and bounded. By the EVT, f does have a max and min. on $x^2+y^2+z^2=2$.

Let $g(x,y,z) = x^2+y^2+z^2$. Note that $\nabla g = (2x, 2y, 2z) = \vec{0}$ only at the origin $(0,0,0)$, which is not on the sphere. So we can disregard this point and move on to solving the Lagrange equations.

$$\nabla f = \lambda g$$

Since $\nabla f = (1, -1, 2)$, we solve $(1,-1,2) = \lambda(2x, 2y, 2z)$.
We have:

Ⅰ $1 = 2\lambda x$
Ⅱ $-1 = 2\lambda y$
Ⅲ $2 = 2\lambda z$
Ⅳ $2 = x^2+y^2+z^2$.

General strategy #2: Solve for x, y, z in terms of λ.

Note that λ cannot be 0, since otherwise eqn (I) turns into $1=0$. Therefore it is safe to divide by λ.

$$\text{(I)}\quad 1 = 2\lambda x \Rightarrow x = \frac{1}{2\lambda}$$
$$\text{(II)}\quad -1 = 2\lambda y \Rightarrow y = \frac{-1}{2\lambda}$$
$$\text{(III)}\quad 2 = 2\lambda z \Rightarrow z = \frac{1}{\lambda}$$

Now (IV) gives $x^2+y^2+z^2=2 \Rightarrow \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{-1}{2\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 = 2$

$$\Rightarrow \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} = 2$$
$$\Rightarrow \frac{3}{2\lambda^2} = 2$$
$$\Rightarrow \lambda^2 = \frac{3}{4}$$
$$\Rightarrow \boxed{\lambda = \pm\frac{\sqrt{3}}{2}}$$

If $\lambda = \frac{\sqrt{3}}{2}$,
$$x = \frac{1}{2\lambda} = \frac{1}{\sqrt{3}}, \quad y = \frac{-1}{2\lambda} = \frac{-1}{\sqrt{3}}, \quad z = \frac{1}{\lambda} = \frac{2}{\sqrt{3}}$$

If $\lambda = -\frac{\sqrt{3}}{2}$,
$$x = \frac{-1}{\sqrt{3}}, \quad y = \frac{1}{\sqrt{3}}, \quad z = \frac{-2}{\sqrt{3}}$$

There are two critical points: $\boxed{\frac{1}{\sqrt{3}}(1,-1,2), \ \frac{1}{\sqrt{3}}(-1,1,-2)}$

$f\left(\frac{1}{\sqrt{3}}(1,-1,2)\right) = \frac{4}{\sqrt{3}}$ global max.

$f\left(\frac{1}{\sqrt{3}}(-1,1,-2)\right) = -\frac{4}{\sqrt{3}}$ global min.

<u>Example 2:</u> Find the point on the ellipse $x^2+y^2=1$ that is closest to the point $(4,-2)$.

<u>Sol:</u> We first must find a fcn to optimize.

Recall that if (x,y) is a point on the plane, the distance from (x,y) to $(4,-2)$ is given by the fcn $d_0(x,y) = \sqrt{(x-4)^2+(y+2)^2}$. We want the point on the constraint $g(x,y) = x^2+y^2=1$ that minimizes d.

Since d_0 is minimal exactly where $d(x,y) = (x-4)^2+(y+2)^2$ we will minimize $d(x,y)$ instead.

Note: ① $x^2+y^2=1$ is closed and bounded, and d is cont. $\Rightarrow$ EVT tells us extrema exist.

② $\nabla g = \vec{0}$ only at the point $(0,0)$ which is not on the ellipse, so we ignore.

We solve: $\nabla f = \lambda \nabla g$

$$\Rightarrow \left(2(x-4), 2(y+2)\right) = \lambda(2x, 2y) \quad \text{and} \quad x^2+y^2=1.$$

$$\Rightarrow \begin{array}{ll} \text{I} & 2(x-4) = 2\lambda x \\ \text{II} & 2(y+2) = 2\lambda y \\ \text{III} & x^2+y^2=1 \end{array}$$

I: Note that if x were 0, I says $2(-4)=0$, which is nonsense. Therefore, we are safe to divide by x since $x \neq 0$. I $\Rightarrow$ $\boxed{\lambda = \frac{x-4}{x}}$.

II: If y were 0, II becomes $(2)(2)=0$, which is nonsense. Therefore, $y \neq 0$ and we are free to divide by y.

$$\text{II} \Rightarrow \boxed{\lambda = \frac{y+2}{y}}$$

As $\lambda = \frac{x-4}{x}$ and $\frac{y+2}{y}$, we have: $\frac{x-4}{x} = \frac{y+2}{y} \Rightarrow$

$$xy - 4y = xy + 2x \Rightarrow \boxed{x = -2y}$$

Now III says $x^2+y^2=1 \Rightarrow (-2y)^2+y^2=1$
$\Rightarrow y = \pm\frac{1}{\sqrt{5}}$

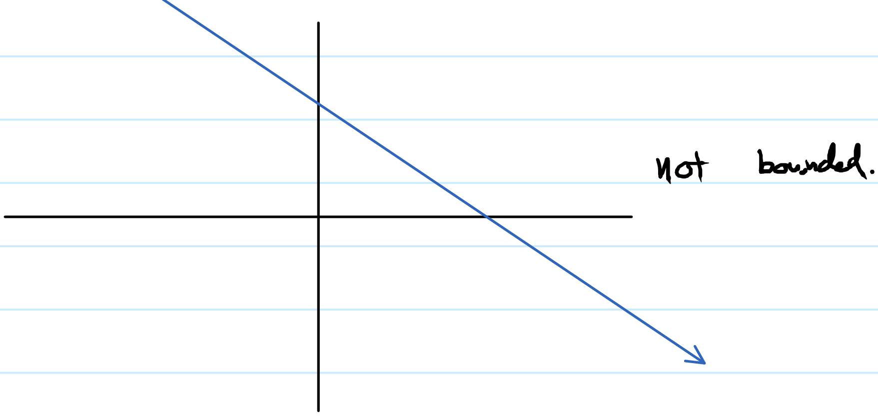
So we get two points: $\left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$.

It is easy to check $\left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$ is the closest to $(4,-2)$ on the unit circle.

Rmk: This answer is obvious because $\left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$ is the normalization of $(4,-2)$.

Example 3: Find the maximum of $f(x,y) = e^{-x^2-y^2}$ the constraint $3y+2x=4$.

Note: The constraint $3y+2x=4$ is a line, and is therefore not bounded. We are not guaranteed to have extrema!



Let's first try to find critical points, and then prove that they do in fact give us the max.

Since $\nabla g = (2,3) \neq \vec{0}$, we move on to solving the Lagrange eqn:

$$\nabla f = \lambda \nabla g$$

$\Rightarrow \quad (-2xe^{-x^2-y^2}, -2ye^{-x^2-y^2}) = \lambda(2,3)$ and $3y+2x=4$.

$\Rightarrow$ I $\quad -2xe^{-x^2-y^2} = 2\lambda$

II $\quad -2ye^{-x^2-y^2} = 3\lambda$

III $\quad 3y+2x=4$

$$\left.\begin{array}{l} \text{I} \Rightarrow \quad \lambda = -xe^{-x^2-y^2} \\ \text{II} \Rightarrow \quad \lambda = -\frac{2}{3}ye^{-x^2-y^2} \end{array}\right\} \Rightarrow xe^{-x^2-y^2} = \frac{2}{3}ye^{-x^2-y^2}$$

As $e^{-x^2-y^2} \neq 0 \Rightarrow \boxed{x = \frac{3}{2}y}$

III $\Rightarrow \quad 3y + 2(\frac{3}{2}y) = 4$

$\Rightarrow y = \frac{2}{3} \Rightarrow x=1$

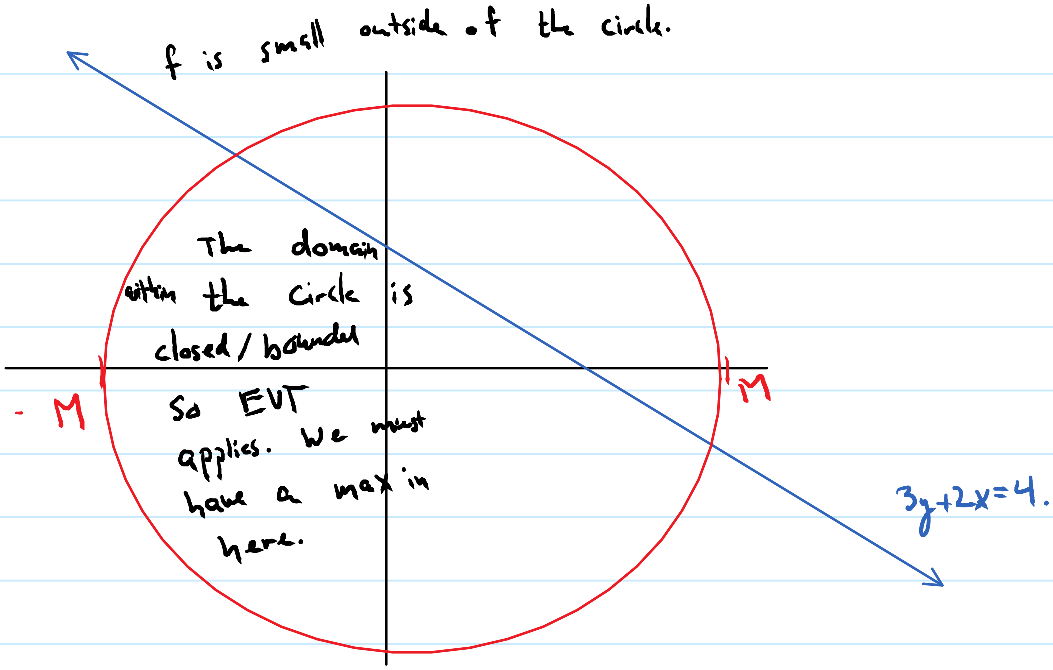
we have one critical point: $\boxed{(1, \frac{2}{3})}$

Now we prove f has a max. at $(1, \frac{2}{3})$.

First note $\quad f(1, \frac{2}{3}) = e^{-1-\frac{4}{9}} = \boxed{e^{-13/9}}$

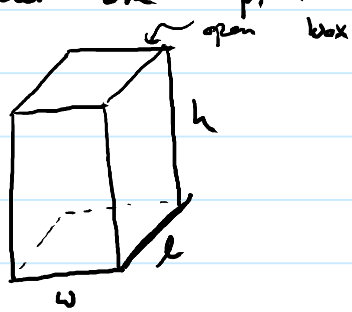
Second, note $\displaystyle\lim_{\|(x,y)\| \to \infty} (e^{-x^2-y^2}) = 0$. I.e., as (x,y) goes to infinity in any direction, $f(x,y) \to 0$. So there is a constant M, so that if $\|(x,y)\| > M$, $f(x,y) < \frac{1}{2}e^{-13/9}$.

Therefore, f restricted to $3y+2x=4$ attains its max within the closed and bounded set $\{(x,y) \mid 3y+2x=4 \ \& \ \|(x,y)\| \leq M\}$. Since $(1, \frac{2}{3})$ was the only point we found, it must actually be the max.



Example 4: What is the maximal area of an open top box with surface area 64 cm^2?

Sol: Consider the picture below:



We want to minimize the fcn $V(w,l,h) = wlh$ given that Surface area $= wl + 2lh + 2wh = 64$.

Let $g(w,l,h) = wl + 2lh + 2wh$.

The Constraint: $wl + 2lh + 2wh = 64$ is unbdd, so it is unclear if a max exists. We show later that we may assume $0 \le w,l,h \le 64$. (See last page).

$\nabla V = (lh,\ wh,\ wl)$
$\nabla g = (l+2h,\ w+2h,\ 2l+2w).$

$\nabla g = \vec{0}$ only at the origin which is not on our constraint.

So we solve: $\vec{\nabla} f = \lambda \vec{\nabla} g$

$$\Rightarrow (lh,\ wh,\ wl) = \lambda(l+2h,\ w+2h,\ 2l+2w)$$

$$\begin{aligned} \Rightarrow \quad \mathrm{I} \quad & lh = \lambda(l+2h) \\ \mathrm{II} \quad & wh = \lambda(w+2h) \\ \mathrm{III} \quad & lw = \lambda(2l+2w) \\ \mathrm{IV} \quad & wl + 2wh + 2lh = 64 \end{aligned}$$

We may assume $w, l, h \neq 0$ since that would give $V(w,l,h) = 0$.

$$\left.\begin{aligned} \mathrm{I} \Rightarrow \lambda &= \frac{lh}{l+2h} \\ \mathrm{II} \Rightarrow \lambda &= \frac{wh}{w+2h} \end{aligned}\right\} \Rightarrow \frac{lh}{l+2h} = \frac{wh}{w+2h}$$

$$\mathrm{III} \Rightarrow \lambda = \frac{lw}{2l+2w}$$

$$\begin{aligned} \Rightarrow \quad & l(w+2h) = w(l+2h) \\ \Rightarrow \quad & \cancel{lw} + 2lh = \cancel{lw} + 2wh \\ \Rightarrow \quad & 2lh = 2wh \\ \Rightarrow \quad & \boxed{l = w} \end{aligned}$$

Now: $\dfrac{wh}{w+2h} = \dfrac{lw}{2l+2w} \Rightarrow \dfrac{wh}{w+2h} = \dfrac{w^2}{4w} = \dfrac{w}{4}$

$$\begin{aligned} \Rightarrow \quad & 4wh = w^2 + 2wh \\ \Rightarrow \quad & 0 = w^2 - 2wh \\ \Rightarrow \quad & 0 = w(w-2h) \end{aligned}$$

$\Rightarrow \cancel{w=0}$ or $w = 2h$

gives Vol $=0$

$\Rightarrow \boxed{l = w = 2h}$

Finally, IV: $wl + 2lh + 2wh = 64$

$\Rightarrow 4h^2 + 4h^2 + 4h^2 = 64$

$\Rightarrow 12h^2 = 64$

$h^2 = \frac{16}{3}$

$\Rightarrow \boxed{h = \frac{4}{\sqrt{3}}}$ (h can't be ≤ 0!)

$\Rightarrow \boxed{w = \frac{8}{\sqrt{3}}, \quad l = \frac{8}{\sqrt{3}}}$

$\Rightarrow$ max. volume $= \left(\frac{8}{\sqrt{3}}\right)\left(\frac{8}{\sqrt{3}}\right)\left(\frac{4}{\sqrt{3}}\right) = \boxed{\frac{256}{3\sqrt{3}}}$

If say $w > 64$,

then we get $64 = wl + 2lh + 2wh \geq wl$

$$\Rightarrow 64 \geq wl$$

Similarly $64 = wl + 2lh + 2wh \geq 2wh$

$$32 > wh$$

$$64 > wl \quad 32 > wh.$$

$$\Rightarrow w^2 l h < 64 \cdot 32 \Rightarrow \text{vol} \leq \frac{64 \cdot 32}{w} < 32$$

Similarly, if $l > 64$, $\text{Vol} \leq 32$,

if $h > 64$, $64 = wl + 2wh + 2lh > 2lh$

$$\Rightarrow 32 > lh$$

similarly, $64 = wl + 2wh + 2lh > 2wh$

$$\Rightarrow 32 > wh$$

$$\Rightarrow 32^2 > wlh^2$$

$$\Rightarrow \frac{32^2}{h} > wlh = \text{Vol}$$

but $\text{vol} < \frac{32^2}{h} < \frac{32^2}{64} < 16.$

So outside of the bounded region, $0 \leq w, l, h \leq 64$, $\text{Vol} \leq 32$.

We see that inside the bounded region $0 \leq w, l, h \leq 64$, we can find a point with larger volume than 32. Therefore that point must actually be a global max.