Longo: Math 20C - Winter 2017
Lecture Notes

Date: January 11, 2017

Section:

- [ ] § 1.1 (cont.)
- [ ] § 1.2

Topics Covered:

- [ ] Special vectors
- [ ] Equations of lines in 3-space
- [ ] Dot product of vectors (also called inner product)

From last time:

A **vector** in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) is a directed line segment in 2-space (resp. 3-space). Two vectors are equivalent if they are translates of each other. This means we only really care about the **length** and **direction** of a vector.

Special Vectors:

The following vectors are especially important:

① The **zero vector**, $\vec{0}$, is the vector that starts and ends at the origin. $\vec{0}$ is the only vector of length zero. In $\mathbb{R}^2$, $\vec{0} = (0,0)$ in $\mathbb{R}^3$, $\vec{0} = (0,0,0)$.

Remark: If $\vec{v}$ is any vector, $\boxed{\begin{array}{l}\vec{0}+\vec{v}=\vec{v}\\ \vec{v}+\vec{0}=\vec{v}.\end{array}}$

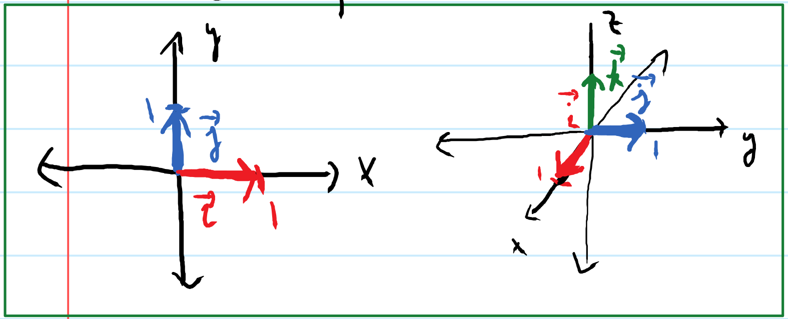
② The vector $\vec{i}$ is the vector of length 1 that points in the positive x-direction

$\vec{i} = (1,0)$ in $\mathbb{R}^2$ and $\vec{i} = (1,0,0)$ in $\mathbb{R}^3$.

③ The vector $\vec{j}$ is the vector of length 1 that points in the positive y-direction

$\vec{j} = (0,1)$ in $\mathbb{R}^2$ and $\vec{j} = (0,1,0)$ in $\mathbb{R}^3$.

④ The vector $\vec{k}$ is the vector of length 1 that points in the positive z-direction $\vec{k} = (0,0,1)$.



$\vec{i}, \vec{j}, \vec{k}$ are important because any vector can be expressed in terms of $\vec{i}, \vec{j}, \vec{k}$.

Ex: ① $(-1,3) = (-1,0) + (0,3) = -1(1,0) + 3(0,1)$

$= -\vec{i} + 3\vec{j}$

② $(6,-2,4) = 6(1,0,0) - 2(0,1,0) + 4(0,0,1) = 6\vec{i} - 2\vec{j} + 4\vec{k}$

$\vec{i}$, $\vec{j}$, $\vec{k}$ are called the standard basis vectors.

Equations of lines in 3-space:

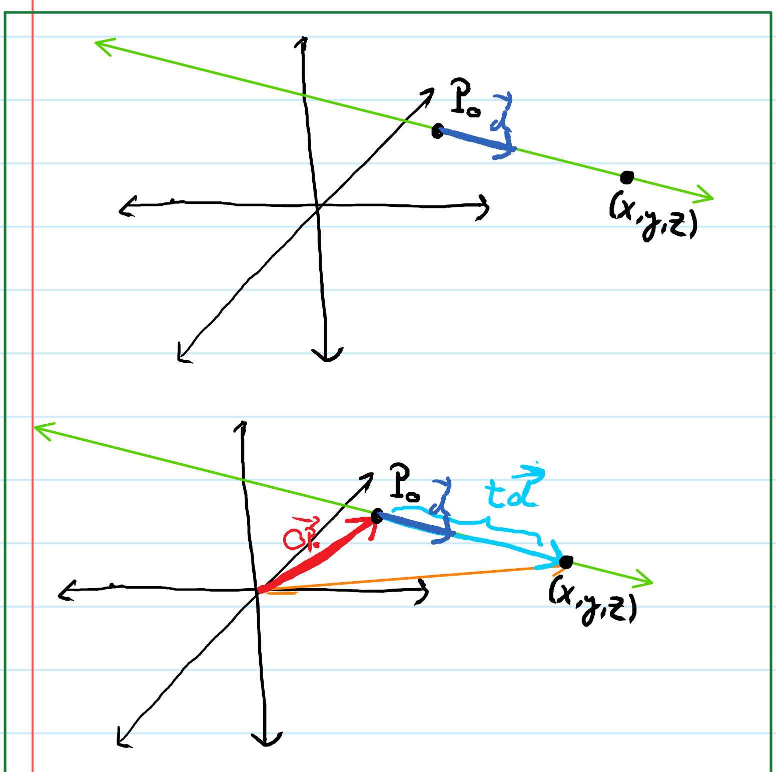
You know that in $\mathbb{R}^2$, you can find the equation of a line if you know: ① a point, (x_0, y_0), on the line, and
② The slope, m, (direction of the line).

Then we can use the point-slope form:

$$(y - y_1) = m(x - x_1).$$

We do something similar in 3-space.

Suppose you know a point $P_0 = (x_0, y_0, z_0)$ on a line, and a vector $\vec{d} = (a, b, c)$ that is parallel to the line.



If (x,y,z) is any other point on the line, we can write $\boxed{(x,y,z) = \vec{OP_0} + t\vec{d}}$ for some real number t.

So any point on the line has the form $\vec{OP_0} + t\vec{d}$ for some t. We say the line can be described parametrically with parameter t, by

vector form

$$\boxed{\begin{aligned} \vec{\ell}(t) &= \vec{OP_0} + t\vec{d} \\ &= (x_0, y_0, z_0) + t(a, b, c) \\ &= (x_0 + ta, y_0 + tb, z_0 + tc) \end{aligned}}$$

Equivalently, the line is described by the parametric equations:

$$\boxed{\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}}$$

Examples: (1) Find the line passing through $(0,9,-1)$ that is parallel to the vector $(6,-2,-1)$.

(2) Find the eqn of the line passing through $(0,9,-1)$ and $(-1,-1,1)$.

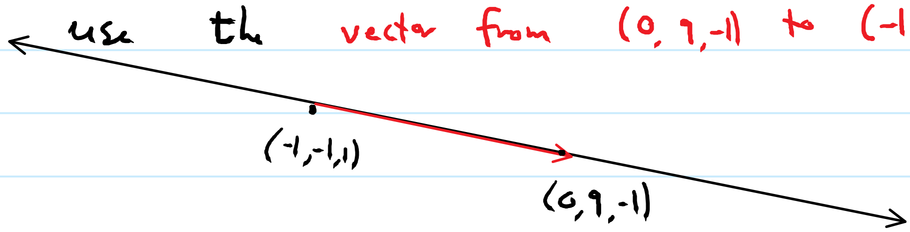
(3) Where do the lines $\vec{l}_1(t) = (1,0,0) + t(2,-1,1)$, $\vec{l}_2(t) = (3,-3,-1) + t(0,1,1)$ intersect?

Sol: (1) Here, $P_0 = (0,9,-1)$, $\vec{d} = (6,-2,-1)$. So the line is

$$\boxed{\vec{l}(t) = (0,9,-1) + t(6,-2,-1)}$$

(2) Recall, we just need any point on the line, and a direction vector, $\vec{d}$.

Let's take $P_0 = (-1,-1,1)$. For $\vec{d}$ we can use the vector from $(0,9,-1)$ to $(-1,-1,1)$.



So take $\vec{d} = (0,9,-1) - (-1,-1,1) = (0+1, 9+1, -1-1) = (1,10,-2)$

So we get $\boxed{\vec{l}(t) = (-1,-1,1) + t(1,10,-2).}$

(3) If the two lines intersect, then the x, y, z components must all be equal. So we set up the equations:

$$(1,0,1) + t(2,-1,1) = (4,-3,0) + s(0,1,1)$$

Rmk: We must change one of the t's to a different variable because they need not intersect at the same "input value".

So we have:

(1) $1 + 2t = 4$

(2) $-t = -3 + s$

(3) $1 + t = s$

From eqn (3) $s = 1+t$, plug this into eqn (2)

$$-t = -3 + (1+t)$$
$$\Rightarrow \quad -2t = -3$$
$$\Rightarrow \quad \boxed{t = \tfrac{3}{2}}$$

Plug this back into ③:
$$\boxed{s = 1 + \tfrac{3}{2} = \tfrac{5}{2}}$$

So the y and z coords are equal when $t = \frac{3}{2}$ and $s = \frac{5}{2}$. Let's plug this back into eqn ① to see if the x-coords are equal:
$$1 + 2\left(\tfrac{3}{2}\right) \overset{?}{=} 4 \quad \checkmark.$$

$\therefore$ They intersect when say $t = \frac{3}{2}$. (or $s = \frac{5}{2}$)

we get $\left(1 + 2\left(\frac{3}{2}\right), -\frac{3}{2}, 1 + \frac{3}{2}\right) = \boxed{\left(4, -\frac{3}{2}, \frac{5}{2}\right)}$

§1.2: The Inner Product:

So far, we discussed how to multiply a vector with a scalar (real number). There are two useful ways to multiply two vectors (neither are what you probably expect). The first way is called the "dot product", or, "inner product".

Def: Let $\vec{u} = (x_1, y_1, z_1)$, $\vec{v} = (x_2, y_2, z_2)$ be two vectors in $\mathbb{R}^3$. The dot product of $\vec{u}$ and $\vec{v}$, denoted $\vec{u} \cdot \vec{v}$, is defined by
$$\boxed{\vec{u} \cdot \vec{v} = x_1 x_2 + y_1 y_2 + z_1 z_2}$$

The dot product of two vectors in $\mathbb{R}^2$ is the same, but you don't include the z-components

Remarks: ① The dot product of two vectors is a number!

② The dot product has nice properties:

For vectors $\vec{u}, \vec{v}, \vec{w}$, and a scalar k,

(i) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (commutativity)

(ii) $(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (k\vec{v})$

(iii) $\vec{u} \cdot (\vec{v} + \vec{w}) = (\vec{u} \cdot \vec{v}) + (\vec{u} \cdot \vec{w})$ (distributivity)

!Warning! DO NOT FORGET THE DOT!!
"$\vec{u}\vec{v}$" has NO meaning.

Example: ① Let $\vec{u} = (0,3,-1)$, $\vec{v} = (5,5,0)$.

$$\vec{u}\cdot\vec{v} = (0)(5) + (3)(5) + (-1)(0)$$
$$= \boxed{15}$$

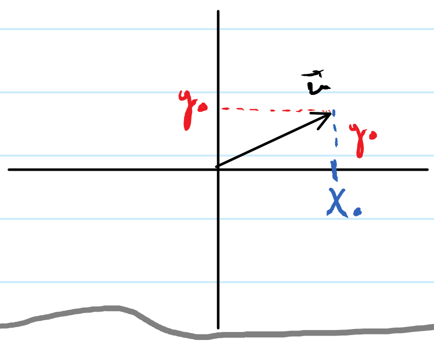
② Let $\vec{a} = (9, 5)$, $\vec{b} = (-5, 9)$

$$\vec{a}\cdot\vec{b} = (9)(5) + (-5)(9)$$
$$= 45 - 45$$
$$= \boxed{0}$$

The dot product and length:

Q? If $\vec{u} = (x_0, y_0)$ is a vector in $\mathbb{R}^2$, what is the length of $\vec{u}$?

Notation: The length of a vector $\vec{u}$ is denoted, $\|\vec{u}\|$. $\|\vec{u}\|$ is also called the norm or magnitude of $\vec{u}$

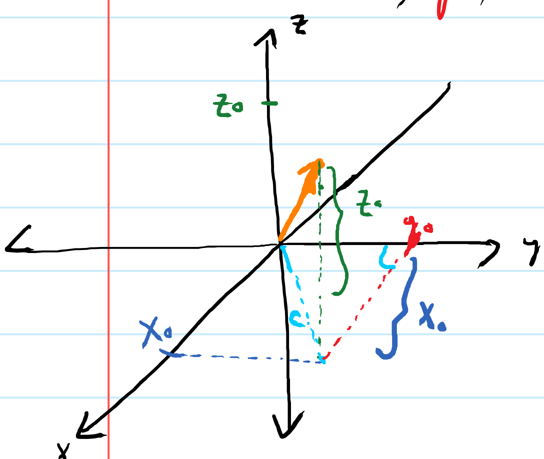


Using Pythagorean thm:

$$\|\vec{u}\|^2 = x_0^2 + y_0^2 \Rightarrow$$
$$\boxed{\|\vec{u}\| = \sqrt{x_0^2 + y_0^2}}$$

What about in $\mathbb{R}^3$?

Let $\vec{v} = (x_0, y_0, z_0)$.



We use Pythagorean Thm twice. In this picture, we look at the right triangle in the xy-plane. The hypotenuse, , has length

$$c = \sqrt{x_0^2 + y_0^2}$$

Now using the Pythagorean thm on the orange, green, l. blue right triangle, we get.

$$\|\vec{v}\|^2 = c^2 + z_0^2$$
$$\boxed{\|\vec{v}\| = \sqrt{x_0^2 + y_0^2 + z_0^2}}$$

Note that if we calculate:

$$\vec{v}\cdot\vec{v} = (x_0, y_0, z_0)\cdot(x_0, y_0, z_0) = x_0^2 + y_0^2 + z_0^2.$$

Therefore. $\boxed{\|\vec{v}\|^2 = \vec{v}\cdot\vec{v}}$ (Important!)

Example: Find the magnitude of $\vec{u} = (1,1,2)$, $\vec{v} = (6,-7)$

Sol: $\|\vec{u}\| = \sqrt{(1)^2+(1)^2+(2)^2} = \sqrt{1+1+4} = \boxed{\sqrt{6}}$

$\|\vec{v}\| = \sqrt{(6)^2+(-7)^2} = \sqrt{36+49} = \boxed{\sqrt{85}}$

Next time: ① Unit vectors / Normalization
② Geometry of the inner product & Orthogonality.
③ Orthogonal Projections.