Longo: Math 20C - Winter 2017
Lecture Notes

Date: March 6, 2017

Section:
- [ ] § 4.1

Topics Covered:
- [ ] Finding position from acceleration
- [ ] Newton's 2nd Law of Motion

§4.1: More on paths and Newton's 2^{nd} Law of Motion:

Recall: If $\vec{c}:\mathbb{R}^n \to \mathbb{R}$ $(n=2 \text{ or } 3)$ is a path fcn

$$\vec{c}(t) = (x(t), y(t)) \text{ or } (x(t), y(t), z(t)),$$

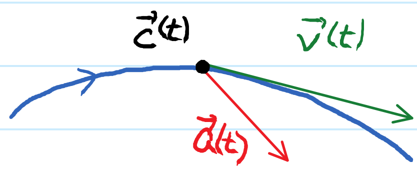
we think of $\vec{c}(t)$ as the position of a particle floating in space at time t. As t progresses, we wish to track the motion of the particle.

To understand the trajectory of the particle, we consider it's velocity vector $\vec{v}(t) = \vec{c}\,'(t) = (x'(t), y'(t))$ or $(x'(t), y'(t), z'(t))$.
we get the Speed of the particle by taking the magnitude of velocity:

$$s(t) = \|\vec{c}\,'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2} \quad \left(= \sqrt{\vec{c}\,'(t)\cdot\vec{c}\,'(t)} \right)$$

Finally, we track the change in velocity by looking at the acceleration vector

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{c}\,''(t)$$



Review: Special Case of differentiation rules:

Let $\vec{c}(t), \vec{r}(t)$ be path fcns. Then

① $\frac{d}{dt}\left((\vec{c}+\vec{r})(t) \right) = \vec{c}\,'(t) + \vec{r}\,'(t)$

② $\frac{d}{dt}\left((\alpha\vec{c})(t) \right) = \alpha\,\vec{c}\,'(t)$

③ $\frac{d}{dt}\left(\vec{c}(t)\cdot\vec{r}(t) \right) = \vec{c}\,'(t)\cdot r(t) + \vec{c}(t)\cdot\vec{r}\,'(t)$

④ $\frac{d}{dt}\left(\vec{c}(t)\times\vec{r}(t) \right) = \vec{c}\,'(t)\times\vec{r}(t) + \vec{c}(t)\times\vec{r}\,'(t).$

Ex: Use the product rule to find $\frac{d}{dt}(\vec{c}(t)\cdot\vec{r}(t))$ where

$$\vec{c}(t) = (t, 1)$$
$$\vec{r}(t) = (\cos(t), \sin(t))$$

$$\frac{d}{dt}(\vec{c}(t)\cdot\vec{r}(t)) = \vec{c}\,'(t)\cdot\vec{r}(t) + \vec{c}(t)\cdot\vec{r}\,'(t)$$
$$= (1,0)\cdot(\cos(t), \sin(t)) + (t,1)\cdot(-\sin(t), \cos(t))$$
$$= \cos(t) - t\sin(t) + \cos(t)$$
$$= \boxed{2\cos(t) - t\sin(t)}$$

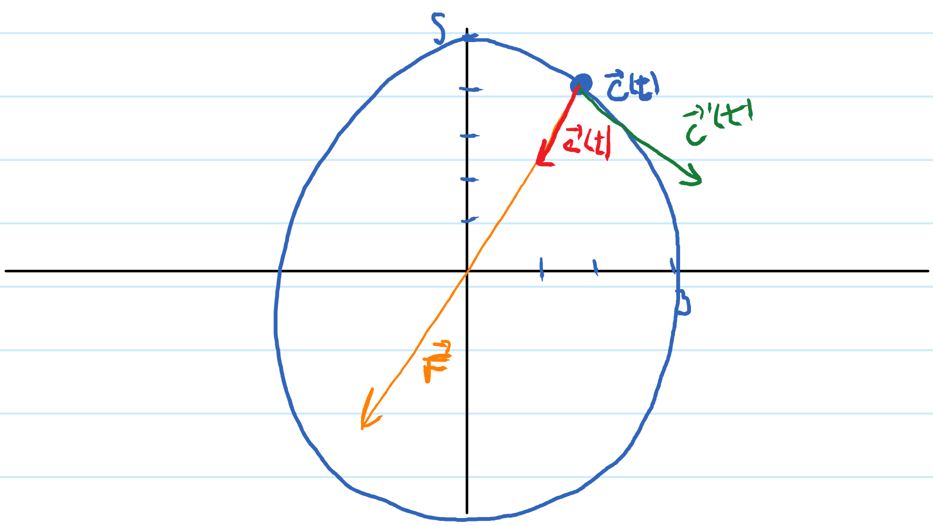
Application: Newton's 2nd Law of Motion:

Force = (mass) x (acceleration)

$$\boxed{\vec{F} = m\vec{a}}$$

Example: A 150 kg mass object travels an elliptical orbit $\vec{c}(t) = (3\sin(t), 5\cos(t))$

(Here distance is meters, time is in sec.)



Calculate the force acting on the object.

Sol: $\vec{c}\,'(t) = (3\cos(t), -5\sin(t))$

$\vec{a}''(t) = (-3\sin(t), 5\cos(t))$

Therefore, the force vector $\vec{F} = m\vec{a}$

$= 150(-3\sin(t), 5\cos(t))$

Equations of motion (vector version):-

In calc. 2, you learned that if you know:

- acceleration
- initial velocity
- initial position

of an object (in a vacuum) then you can fully describe the motion of the object. We can do the same with vector valued fcns. For example, since $\vec{c}\,'(t) = \vec{v}(t)$ for some path fcn $\vec{c}(t)$, we can integrate the components of $\vec{v}(t)$ to recover $\vec{c}(t)$ (up to a constant). If we also know some initial value of $\vec{c}$, you can determine $\vec{c}$ uniquely.

Example: Let $\vec{c}(t)$ be a path fcn. Suppose

- $\vec{v}(t) = (t, t^2, 0)$
- $\vec{c}(1) = (0, 3, 1)$

① Find a formula for $\vec{c}(t)$.

② Evaluate $\vec{c}(5)$.

③ Find the displacement vector, $\vec{c}(5) - \vec{c}(1)$, time $t=1$ to $t=5$.

Sol: ① Since $\vec{c}\,'(t) = \vec{v}(t) = (t, t^2, 0)$ we must have

$$\vec{c}(t) = \left(\int t\,dt, \int t^2 dt, \int 0\,dt\right) = \left(\tfrac{1}{2}t^2 + C_1,\ \tfrac{1}{3}t^3 + C_2,\ C_3\right)$$

for some constants C_1, C_2, C_3. We can write this more succinctly as $\vec{c}(t) = \left(\tfrac{1}{2}t^2, \tfrac{1}{3}t^3, 0\right) + \vec{u}$ where $u = (C_1, C_2, C_3)$ is a constant vector.

to find $\vec{u}$ we use the fact that $\vec{c}(1) = (0,3,1)$. I.e, if we plug in $t=1$, we get $(0,3,1)$.

$$\therefore \quad (0,3,1) = \vec{c}(1) = \left(\tfrac{1}{2}(1)^2, \tfrac{1}{3}(1)^3, 0\right) + \vec{u}$$

$$\Rightarrow \quad (0,3,1) = \left(\tfrac{1}{2}, \tfrac{1}{3}, 0\right) + \vec{u}$$

$$\Rightarrow \quad \vec{u} = (0,3,1) - \left(\tfrac{1}{2}, \tfrac{1}{3}, 0\right) = \left(-\tfrac{1}{2}, \tfrac{8}{3}, 1\right)$$

So $\boxed{\vec{c}(t) = \left(\tfrac{1}{2}t^2, \tfrac{1}{3}t^3, 0\right) + \left(-\tfrac{1}{2}, \tfrac{8}{3}, 1\right)}$

Rmk: Alternatively, we could use the 2nd fundamental Thm of calculus:

$$\vec{c}(t) - \vec{c}(1) = \int_1^t \vec{c}\,'(x)\,dx$$

$$\vec{c}(t) - \vec{c}(1) = \int_1^t (x, x^2, 0)\,dx$$

$$\vec{c}(t) - \vec{c}(1) = \left(\tfrac{1}{2}x^2, \tfrac{1}{3}x^3, 0\right)\Big|_{x=1}^{x=t}$$

$$\vec{c}(t) - (0,3,1) = \left(\tfrac{1}{2}t^2, \tfrac{1}{3}t^3, 0\right) - \left(\tfrac{1}{2}, \tfrac{1}{3}, 0\right)$$

$$\vec{c}(t) = \left(\tfrac{1}{2}t^2, \tfrac{1}{3}t^3, 0\right) - \left(\tfrac{1}{2}, \tfrac{1}{3}, 0\right) + (0,3,1)$$

$$\boxed{= \left(\tfrac{1}{2}t^2, \tfrac{1}{3}t^3, 0\right) + \left(-\tfrac{1}{2}, \tfrac{8}{3}, 1\right)}$$

(2) $\vec{c}(3) = \left(\tfrac{9}{2}, 9, 0\right) + \left(-\tfrac{1}{2}, \tfrac{8}{3}, 1\right)$
$= \left(4, \tfrac{35}{3}, 1\right)$

(3) $\vec{c}(3) - \vec{c}(1) = \left(4, \tfrac{35}{3}, 1\right) - (0,3,1) = (4, 13, 0)$

So the particle went 4 units in the positive x-dir.
13 units in positive y-dir.
0 units in z-dir.

Remark: if we only wanted part (c), we could do

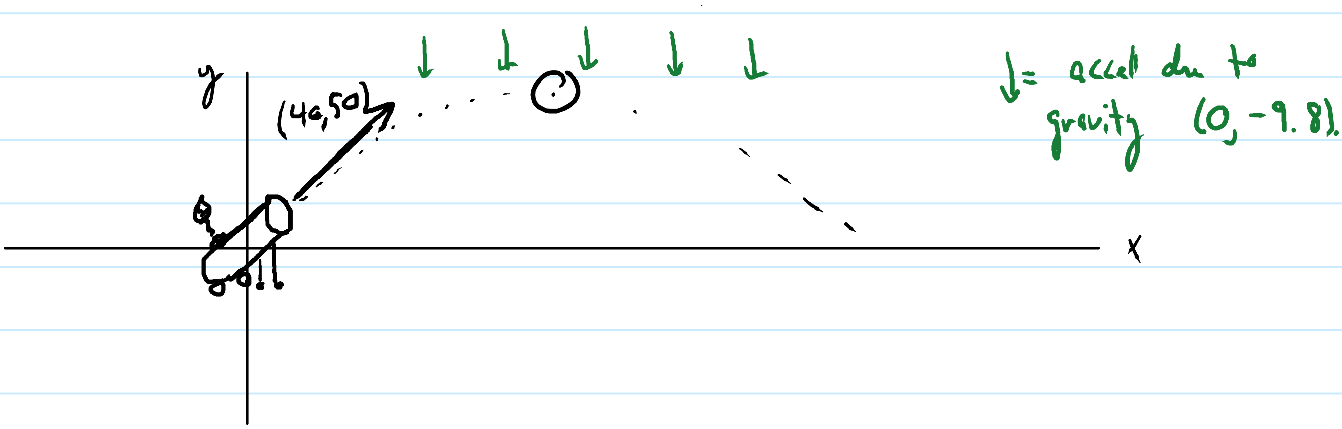
$$\vec{c}(3) - \vec{c}(1) = \int_1^3 \vec{c}\,'(t)\,dt$$

$$\vec{c}(3) - \vec{c}(1) = \int_1^3 (t, t^2, 0)\,dt$$

$$= \left(\tfrac{1}{2}t^2, \tfrac{1}{3}t^3, 0\right)\Big|_{t=1}^{t=3} = \left(\tfrac{1}{2}(3)^2, \tfrac{1}{3}(3)^3, 0\right) - \left(\tfrac{1}{2}, \tfrac{1}{3}, 0\right)$$

$$= \left(\tfrac{9}{2}, \tfrac{27}{3}, 0\right) - \left(\tfrac{1}{2}, \tfrac{1}{3}, 0\right) = \left(4, 13, 0\right)$$

Ex: A cannonball is shot out of a canon at the point (0,0). At $t=0$ seconds, it's velocity is given by $(50,50)$ (Here the units are m/s). Find the fcn that describes the motion of the ball.



here, $\vec{a}(t) = (0, -9.8)$ (acel. due to gravity)

$$\vec{v}(t) = \left(\int 0\,dt, \int -9.8\,dt \right)$$
$$= (0, -9.8t) + \vec{v}_0$$ where $\vec{v}_0$ is a constant vector.

Since $\vec{v}(0) = (40,50)$, we have
$$(40,50) = (0,0) + \vec{v}_0 \Rightarrow \vec{v}_0 = (40,50).$$

$\Rightarrow$ $\vec{v}(t) = (40, -9.8t + 50)$. If $\vec{c}(t)$ is the position vector,
$\vec{c}(t) = (40t, -4.9t^2 + 50t) + \vec{c}_0$. where $\vec{c}_0$ is a constant vector.

Since $\vec{c}(0) = (0,0)$, we have

$$(0,0) = \vec{c}(0) = (0,0) + \vec{c}_0 \Rightarrow \vec{c}_0 = (0,0)$$

$$\Rightarrow \boxed{\vec{c}(t) = (40t, -4.9t^2 + 50t).}$$