Longo: Math 20C - Winter 2017
Lecture Notes

Date: March 8, 2017

Section:
- [ ] §4.2

Topics Covered:
- [ ] Arc Length

§4.2: Arc Length Formula:

Q? If you drive in a straight line going 60 mph for 4 hours, how far did you travel?

Sol: I'll give you a moment to think,

...,

..,

...

Okay, so the total distance travelled is

speed×time: $(60 \text{ m/h})(4\text{h}) = 240\text{m}$.

More generally, you probably learned in Calc. 2, that if you integrate speed over some time interval, you get the total distance travelled over that time interval.

So our intuition tells us:

Thm: Let $\vec{c}(t)$ be a diff'ble path fcn defined on the (time) interval $a \le t \le b$, then the (arclength / total distance travelled) is given by

$$\int_a^b (\text{speed})\,dt = \int_a^b \|\vec{c}\,'(t)\|\,dt$$

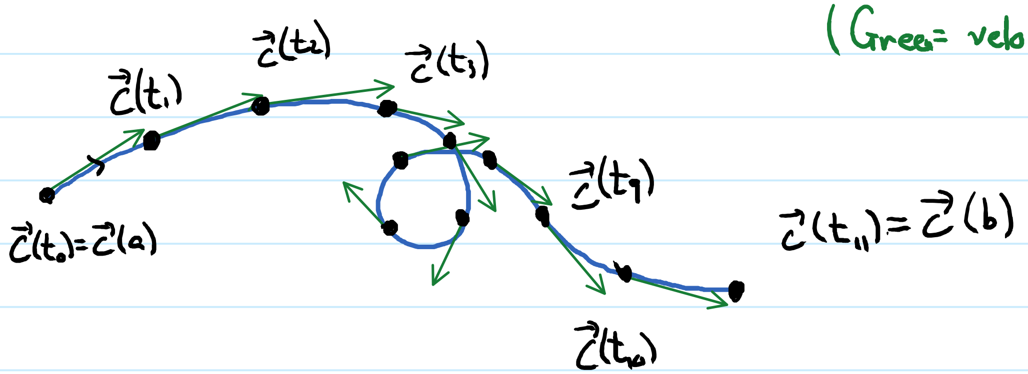
$$= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt \quad (\text{in } \mathbb{R}^2)$$

$$\left(\text{or} \quad = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt \quad (\text{in } \mathbb{R}^3)\right)$$

Informal "proof": Let $\vec{c}(t)$ be a diff'ble path fcn defined on the time interval $a \leq t \leq b$.

Idea: Split $[a,b]$ into N many small pieces, of equal length and for $i = 0, 1, \dots, N$, let t_i be the endpt. of the i^{th} subinterval:

(Green = velocity vectors)



the arclength traced by $\vec{c}$ on the time subinterval $[t_i, t_{i+1}]$ can be approximated by:

$$\begin{aligned} & (\text{speed at time } t_i) \times (\text{time elapsed}) \\ = \; & s(t_i) \times (t_{i-1} - t_i) \\ = \; & s(t_i)\,\Delta t \end{aligned}$$

(See green text for details)

where $s(t_i) = \|\vec{c}\,'(t_i)\|$, $\Delta t = t_{i-1} - t_i$.

So
$$\text{arclength} = \sum_{i=1}^{N} (\text{arclength on the } i^{th} \text{ subinterval}) \approx \sum_{i=1}^{N} \|\vec{c}\,'(t_i)\|\,\Delta t.$$

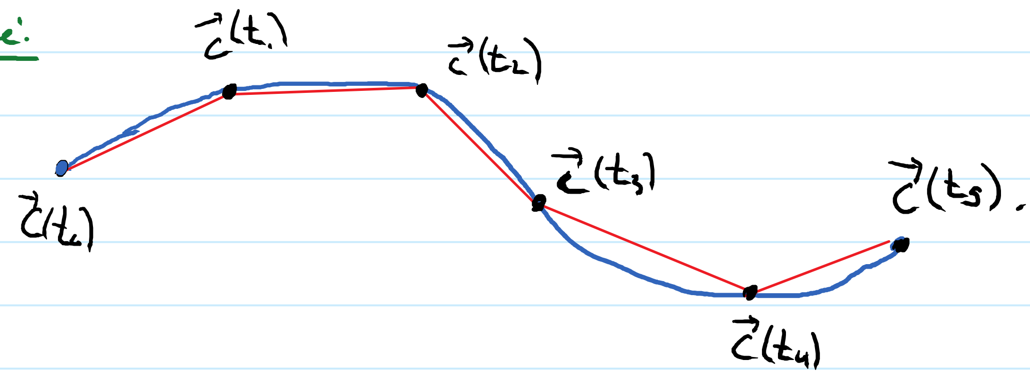
As the number of subintervals, $N \to \infty$, the estimate is more and more accurate. Therefore,

$$\text{arclength} = \lim_{N \to \infty} \left(\sum_{i=1}^{N} \|\vec{c}\,'(t)\| \Delta t \right)$$

This is a limit of Riemann Sums, which is the definition of the (definite) integral.

$$\Rightarrow \quad \text{arclength} = \lim_{N\to\infty}\left(\sum_{i=1}^{N} \|\vec{c}\,'(t)\|\,\Delta t\right)$$
$$= \int_a^b \|\vec{c}\,'(t)\|\,dt.$$

Picture:



Approximate arclength on each subinterval by a straight line. For the i^{th} subinterval, the length of the line segment is

$$\|\vec{c}(t_{i+1}) - \vec{c}(t_i)\| = \left(\frac{\|\vec{c}(t_{i+1}) - \vec{c}(t_i)\|}{t_{i+1} - t_i}\right)(t_{i+1} - t_i).$$

If $\Delta t := t_{i+1} - t_i$ is small (i.e., # of subintervals is large), then

$$\|\vec{c}(t_{i+1}) - \vec{c}(t_i)\| = \left(\frac{\|\vec{c}(t_{i+1}) - \vec{c}(t_i)\|}{\Delta t}\right)\Delta t$$

$$= \|\vec{c}\,'(t_*)\|\,\Delta t$$

where t_* is a # s.t. $t_i \leq t_* \leq t_{i+1}$ (here we used mean value thm).

Example: ① Find the length of the path traced by the fcn $\vec{c}(t) = (4\cos(2t), 4\sin(2t))$ from $t=0$ to $t=2\pi$.

Sol: Rmk: $\vec{c}(t)$, $0 \le t \le 2\pi$ travels the circumference of a circle of radius 4 twice. We should expect arclength to be $2 \times (\text{circumference}) = (2)(8\pi) = 16\pi$.

$$\vec{c}\,'(t) = (-8\sin(2t), 8\cos(2t)).$$

$$\|\vec{c}\,'(t)\| = \left((-8\sin(2t))^2 + (8\cos(2t))^2\right)^{1/2}$$
$$= (64\sin^2(2t) + 64\cos^2(2t))^{1/2}$$
$$= (64)^{1/2}$$
$$= 8.$$

$$\text{Arclength} = \int_0^{2\pi} 8\,dt = 8t\Big|_0^{2\pi}$$
$$= 16\pi - 0$$
$$= \boxed{16\pi}$$

② Find the length of the path travelled by

$$\vec{c}(t) = \left(\ln(\sqrt{t}), -\sqrt{3}\,t, \tfrac{3}{2}t^2\right) \quad \text{from } t=1 \text{ to } t=2.$$

Sol: $\vec{c}\,'(t) = \left(\left(\frac{1}{\sqrt{t}}\right)\left(\frac{1}{2\sqrt{t}}\right), \sqrt{3}, 3t\right)$

$$= \left(\frac{1}{2t}, \sqrt{3}, 3t\right).$$

$$\|\vec{c}\,'(t)\| = \left(\left(\frac{1}{2t}\right)^2 + (\sqrt{3})^2 + (3t)^2\right)^{1/2}$$
$$= \left(\frac{1}{4t^2} + 3 + 9t^2\right)^{1/2}$$

$$= \left(\frac{1 + 12t^2 + 36t^4}{4t^2}\right)^{1/2}$$

$$= \left(\frac{(6t^2+1)^2}{4t^2} \right)^{1/2}$$

$$= \frac{6t^2+1}{2t} = \frac{6t^2}{2t} + \frac{1}{2t}$$

$$= \boxed{3t + \frac{1}{2t}}$$

$$\Rightarrow \quad \text{arclength} = \int_1^2 \|\vec{c}\,'(t)\|\,dt$$

$$= \int_1^2 3t + \frac{1}{2t}\,dt$$

$$= \left(\frac{3}{2}t^2 + \frac{1}{2}\ln|t| \right)\Big|_1^2$$

$$= \left(\frac{3}{2}(2)^2 + \frac{1}{2}\ln(2) \right) - \left(\frac{3}{2} + \frac{1}{2}\cancel{\ln(1)}^{\,0} \right)$$

$$= \left(6 + \frac{1}{2}\ln(2) - \frac{3}{2} \right)$$

$$= \frac{9}{2}\boxed{+ \frac{1}{2}\ln(2)}$$