# Longo: Math 20C - Winter 2017
# Lecture Notes

Date: March 9, 2017

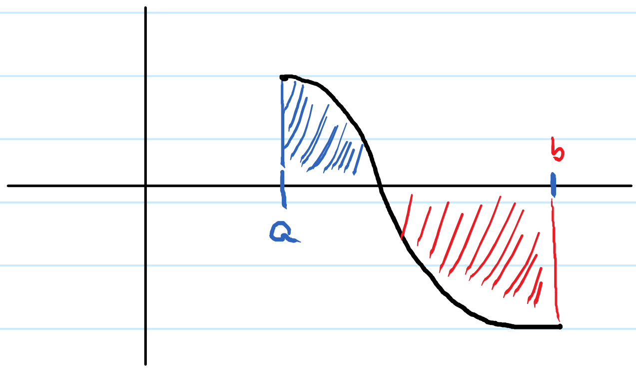
Section:
- [ ] § 5.1

Topics Covered:
- [ ] Volume and Cavalieri's Principle
- [ ] Double integration as iterated integrals
- [ ] Fubini's Theorem

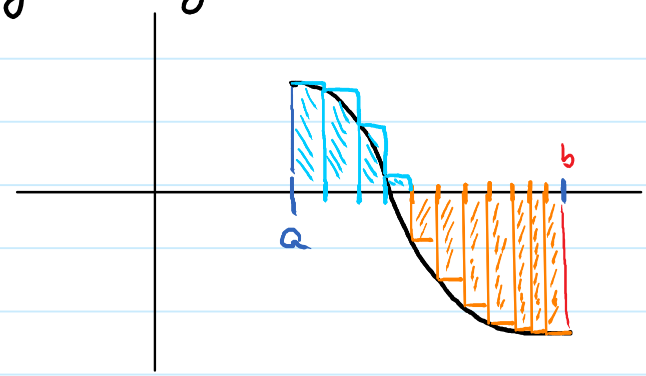
# §5.1: Double Integrals as Volume & Cavalieri's Principle:

If $f(x)$ is a continuous fcn on the closed interval $[a,b]$, you learned that the (signed) area between the graph of $f$ and the x-axis is given by the definite integral

$$\int_a^b f(x)\,dx = (\text{blue area}) - (\text{red area})$$




Recall that in order to define the integral, we slice up the interval $[a,b]$ into N many equal parts, and approximate the area by rectangles

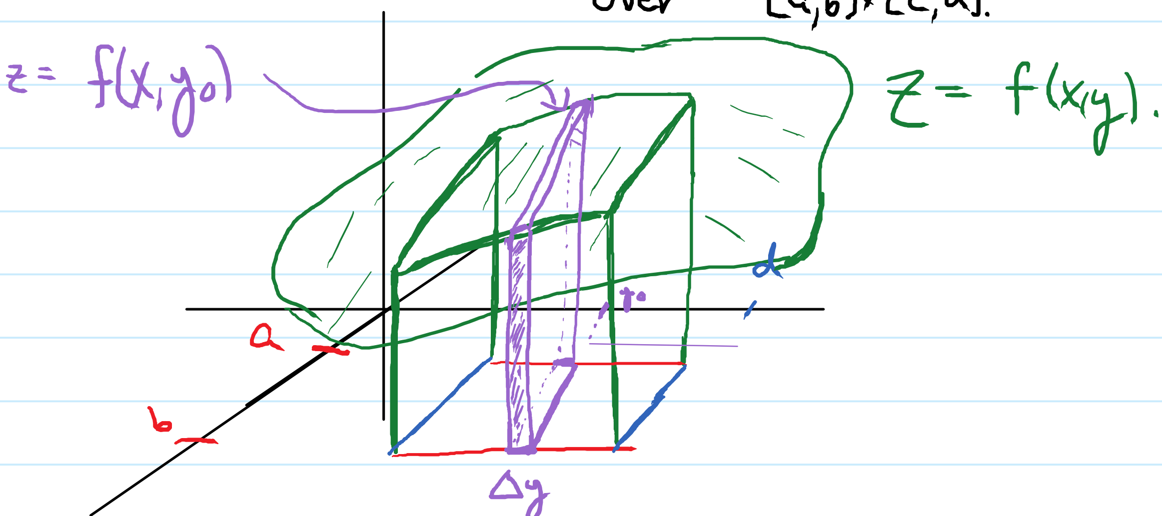



Then the integral is roughly the sum of the areas of the rectangles. As $N \to \infty$, the approximation $\underbrace{\sum_{i=1}^{N} f(x_i)\Delta x}_{\text{Right hand Riemann Sum}} \to \int_a^b f(x)\,dx$

This process is an example of Cavalieri's Principle: In order to find area/volume of a region, we slice it into pieces, find the area/volume of each slice and add them all up. As the number of slices $\to \infty$, we get the area/volume exactly.

Let $f(x,y)$ be a continuous fcn defined on a rectangle $[a,b]\times[c,d] = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$.
($[a,b]\times[c,d]$ is called the Cartesian Product of $[a,b]$ and $[c,d]$.)

We wish to find the (signed) volume of the region between the graph of f and the xy-plane over $[a,b]\times[c,d]$.




If we fix the $y$-variable ($y=y_0$) and slice the volume into pieces that are $\parallel$ to the xz-plane, we can approximate the volume of the slice by

$$(\Delta y)\times(A(y))$$

where $A(y)$ is the area of the face of the slice. If M is the number of slices, then

$$\mathrm{Vol} = \underbrace{\lim_{M\to\infty}\left(\sum_{i=1}^{M} A(y)\Delta y\right)}_{\text{definition of definite integral}} = \int_c^d A(y)\,dy$$

On the other hand, for any fixed $y_0$,

$A(y_0)$ is the area under the graph of the fcn $f(x,y_0)$. Therefore,

$$A(y_0) = \int_a^b f(x,y_0)\,dx$$ (integral w.r.t. x. y is considered a constant).

All together, Volume under $z = f(x,y)$ =

$$\int_c^d A(y)\,dy = \int_c^d \left( \int_a^b f(x,y)\,dx \right) dy. \overset{\text{notation}}{=} \iint_{[a,b]\times[c,d]} f(x,y)\,dA$$

This is called the double integral.

In plain english, to get the volume, integrate $f(x,y)$ w.r.t. x from $x=a$ to $x=b$ (pretend y is constant). Then take the result, and integrate w.r.t. y from $y=c$ to $y=d$.

Ex. Find the volume under the paraboloid $z = x^2+y^2$ over the rectangle $[0,2]\times[-1,1]$.

Sol: Here, $0 \le x \le 2$, $-1 \le y \le 1$.

So

$$\text{Vol.} = \int_{-1}^{1}\int_0^2 x^2+y^2\,dx\,dy$$

$$= \int_{-1}^{1} \left( \tfrac{1}{3}x^3 + y^2 x \right)\Big|_{x=0}^{x=2} dy$$

$$= \int_{-1}^{1} \left(\tfrac{8}{3} + 2y^2\right) - \left(0 + 0\cdot y^2\right) dy = \int_{-1}^{1} \tfrac{8}{3} + 2y^2\,dy$$

$$= \left(\frac{8}{3}y + \frac{2}{3}y^3\right)\Big|_{y=-1}^{y=1}$$

$$= \left(\frac{8}{3} + \frac{2}{3}(1)^3\right) - \left(-\frac{8}{3} + \frac{2}{3}(-1)^3\right)$$

$$= \frac{10}{3} - \left(-\frac{10}{3}\right)$$

$$= \boxed{\frac{20}{3}}$$

**Rmk:** In the initial discussion, we could have fixed $x$ first, and made slices $\parallel$ to the $yz$-plane. In the end, we would still get the same answer. This gives us:

**Thm: (Fubini):** If $f$ is cont. on $[a,b]\times[c,d]$

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

In other words, we can switch the order of integration.

**Ex:** Compute $\displaystyle\int_0^1\int_0^2 \frac{x}{1+xy}\,dx\,dy$.

As written, we would need to compute the "inside" integral ($y$ is "constant")

$\displaystyle\int_0^2 \frac{x}{1+xy}\,dx$. We could try to substitute:

$$u = 1 + xy$$

$$\Rightarrow du = y\,dx \quad \text{and} \quad x = \frac{u-1}{y}$$

$$\Rightarrow \int \frac{x}{1+xy}\,dx = \int \left(\frac{u-1}{y}\right)\left(\frac{1}{u}\right)\left(\frac{1}{y}\right)du.$$

This is one option... but it doesn't look fun instead, let's ignore our problems for now and change the order of integration. Pay attn to bounds!

$$\int_0^1\int_0^2 \frac{x}{1+xy}\,dx\,dy = \int_0^2\int_0^1 \frac{x}{1+xy}\,dy\,dx$$

Now the "inside integral" is

$$\int_0^1 \frac{x}{1+xy}\,dy$$ ( $x$ is "constant")

Let $u = 1+xy \Rightarrow \frac{du}{dy} = x$

$\Rightarrow du = \underline{x\,dy}$

This shows up in the integral! OH BABY!

Then $$\int_0^1 \frac{x}{1+xy}\,dy = \int_{u(0)}^{u(1)} \frac{1}{u}\,du = \ln|u|\Big|_{u(0)}^{u(1)}$$

$$= \ln|1+xy|\Big|_{y=0}^{y=1}$$

$$= \ln|1+x| - \ln(1)$$

$$= \ln|1+x|.$$

Now $$\int_0^2\int_0^1 \frac{x}{1+xy}\,dy\,dx = \int_0^2 \ln(1+x)\,dx$$

$$\overset{*}{=} (1+x)\ln(1+x) - (1+x)\Big|_0^2$$

$$= (3\ln(3) - 3) - ((1)\ln(1) - 1)$$

$$= 3\ln(3) - 3 + 1$$

$$= \boxed{3\ln(3) - 2}$$

(*)

sub $\boxed{\begin{array}{l} w = x+1 \\ dw = dx \end{array}}$

$$\int \ln(x+1)\,dx = \int \ln(w)\,dw$$

Integration by parts

$\boxed{\begin{array}{ll} u = \ln w & dv = dw \\ du = \frac{1}{w}\,dw & v = w \end{array}}$

$$\int \ln(w)\,dw = w\ln(w) - \int dw = w\ln(w) - w$$

$$= \underline{(x+1)\ln(x+1)} - \underline{(x+1)}$$