Longo: Math 20C - Winter 2017
Lecture Notes

Date: March 13, 2017

Section:

- [ ] §5.2
- [ ] §5.3 (intro)

Topics Covered:

- [ ] The definition of the double integral
- [ ] Fubini's Theorem
- [ ] Introduction to integrals over more general regions

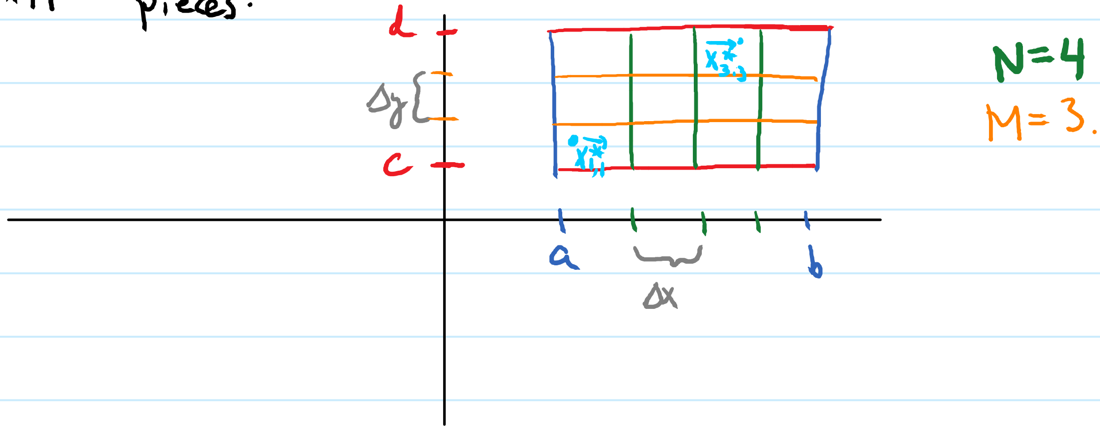
§ 5.2 Definition of the double integral using Riemann Sums:

Fix a rectangle $R=[a,b]\times[c,d]$ in $\mathbb{R}^2$. Let $f:R\subseteq\mathbb{R}^2\to\mathbb{R}$ be continuous. We saw last time that

$$\iint_R f\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy \underset{\text{Fubini}}{=} \int_a^b\int_c^d f(x,y)\,dy\,dx$$

Represents the (signed) volume under the graph of $f(x,y)$.
We viewed these as iterated integrals.
Today we give a Rigorous definition via Riemann Sums:

Partition the interval $[a,b]$ into N pieces and partition $[c,d]$ into M pieces, of equal size.
this gives us a partition of the rectangle R into $N\times M$ pieces:



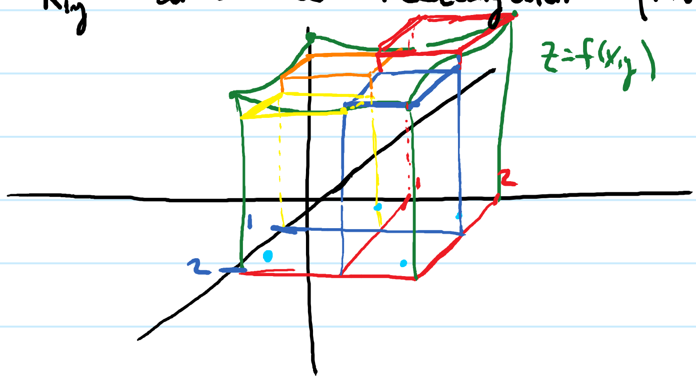
Let $R_{i,j}$ be the Cartesian Product of the i^{th} subinterval of $[a,b]$ with the j^{th} subinterval of $[c,d]$.
Note: R_{ij} has area $\Delta x\,\Delta y$
Let $\vec{x}^*_{ij}$ be an arbitrary fixed point in R_{ij}.
Then define

$$S_{N,M}=\sum_{j=1}^{M}\sum_{i=1}^{N} f(\vec{x}^*_{ij})\,\Delta x\,\Delta y$$

$S_{N,M}$ is called a Riemann Sum.

What is $S_{N,M}$ geometrically? For each i,j, $\Delta x \Delta y = \text{Area}(R_{ij})$.

So $f(\vec{x}_{ij}^*)\Delta x \Delta y$ is the volume of the rectangular prism with base R_{ij} and height $f(\vec{x}_{ij}^*)$. I.e., above R_{ij} draw a rectangular prism of height $f(\vec{x}_{ij}^*)$.



Then $S_{N,M}$ = sum of volumes of rectangular prisms

is an estimate for the volume under the graph via boxes. This is analagous to estimating the area under the curve of a one variable fcn via rectangles.

As $N, M \longrightarrow \infty$, the partition gets finer, and the approximation gets better and better. Therefore, if we take limits, we get the double integral:

$$\iint_R f(x,y)\,dA \overset{\text{definition}}{=} \lim_{\substack{N\to\infty \\ M\to\infty}} \sum_{j=1}^{M}\sum_{i=1}^{N} f(\vec{x}_{ij}^*)\,\Delta x \Delta y$$

Remark: It is easy to see

$$\sum_{j=1}^{M}\sum_{i=1}^{N} f(x,y)\,\Delta x \Delta y = \sum_{i=1}^{N}\sum_{j=1}^{M} f(\vec{x}_{ij}^*)\,\Delta y\,\Delta x$$

I.e., in Riemann Sums, we can switch the

Order of Summation. Thus, Fubini's Thm shouldn't be surprising

The limit definition is typically very difficult and impractical to use. However, <u>Fubini's theorem</u> also states that if f is continuous on the (closed bounded) domain D, then

$$\lim_{\substack{N\to\infty \\ M\to\infty}} \sum_{j=1}^{M}\sum_{i=1}^{N} f(\vec{x}_{ij}^*)\,\Delta x\,\Delta y = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy.$$

iterated integral

So we can just do iterated integrals instead of the limit def.

<u>Examples:</u> ① Let $R=[0,1]\times[0,2]$. Calculate $\iint_R y e^{xy}\,dA$.

If we try to integrate w.r.t. y first, we would have to integrate by parts. That sounds awful. Let's integrate w.r.t. x first.

$$\int_0^2\int_0^1 y e^{xy}\,dx\,dy = \int_0^2\left[\left(\frac{y}{y}e^{xy}\right)\Big|_{x=0}^{x=1}\right]dx = \int_0^2 e^{y}-e^{0}\,dy$$

$$= \int_0^2 e^{y}-1\,dy$$

$$= (e^{y}-y)\Big|_{y=0}^{y=2}$$

$$= (e^2-2)-(e^0-0)$$

$$= \boxed{e^2-3}$$

<u>Warning (common mistake):</u> Double integrals don't generally split up under mult.

$$\boxed{\iint y e^{xy}\,dx\,dy \neq \left(\int y\,dy\right)\left(\int e^{xy}\,dx\right)}$$

② $\iint_R \frac{y^2}{1+x^2}\,dA$ where $R=[0,1]\times[-1,1]$.

$$\int_0^1\int_{-1}^1 \frac{y^2}{1+x^2}\,dy\,dx = \frac{1}{3}\int_0^1\left(\frac{y^3}{1+x^2}\right)\Big|_{-1}^{1}dx = \frac{1}{3}\int_0^1 \frac{(1)^3-(-1)^3}{1+x^2}\,dx$$

$$= \frac{1}{3}\int_0^1 \frac{2}{1+x^2}\,dx$$

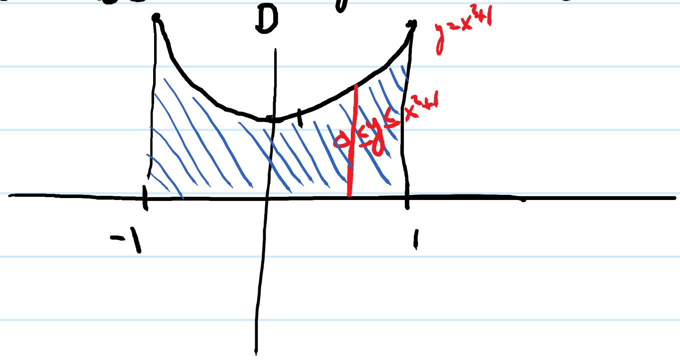
$$= \frac{2}{3}\int_0^1 \frac{1}{1+x^2}\,dx$$

$$= \frac{2}{3}\left(\tan^{-1}(x)\right)\Big|_{x=0}^{1}$$

$$\frac{2}{3}\left(\tan^{-1}(1)-\tan^{-1}(0)\right)$$
$$=\frac{2}{3}\left(\frac{\pi}{4}-0\right)$$
$$=\frac{\pi}{6}$$

§5.3: Integrals over more general Regions!

So far, we've only integrated over rectangles. Let's suppose we want to find the volume of the region in under the paraboloid $z=1-x^2-y^2$ and above the region $D=\{(x,y)\mid {-1\le x\le 1 \atop 0\le y\le x^2+1}\}$. Then we want to calculate $\iint_D 1-x^2-y^2\,dA$ where:



On D, we notice that we can find bounds for y in terms of x: $0\le y\le x^2+1$. Since $-1\le x\le 1$ on D, we can set up an iterated integral:

$$\underbrace{\int_{-1}^{1}\underbrace{\int_0^{x^2+1}}_{\text{bounds for } y}}_{\text{bounds for } x} 1-x^2-y^2\,dy\,dx \Rightarrow \text{Inner integral} = \int_0^{x^2+1} 1-x^2-y^2\,dy$$

$$=\left(y-x^2y-\frac{1}{3}y^3\right)\Big|_{y=0}^{y=x^2+1}$$

$$=(x^2+1)-x^2(x^2+1)-\frac{1}{3}(x^2+1)^3$$

$$=\cancel{x^2}+1-x^4-\cancel{x^2}-\frac{1}{3}(x^6+3x^4+3x^2+1)$$

$$=-\frac{1}{3}x^6-2x^4-x^2+\frac{2}{3}$$

Note: $\int_0^{x^2+1} f(x,y)\,dy$ = Area of slice with fixed x value

So $$\int_{-1}^{1}\int_0^{x^2+1} 1-x^2-y^2\,dy\,dx=\int_{-1}^{1}-\frac{1}{3}x^6-2x^4-x^2+\frac{2}{3}\,dx$$

$$= 2\int_0^1 -\frac{1}{3}x^6 - 2x^4 - x^2 + \frac{2}{3}\,dx$$

$$= 2\left(-\frac{1}{21}x^7 - \frac{2}{5}x^5 - \frac{1}{3}x^3 + \frac{2}{3}x\right)\Big|_0^1$$

$$= 2\left(-\frac{1}{21} - \frac{2}{5} - \frac{1}{3} + \frac{2}{3}\right)$$

$$= 2\left(\frac{-4}{35}\right)$$

$$= \frac{-8}{35}$$

integrating an even fcn!

Warning: ① The bounds on the "outer integral" should never have one of the variables. (We have to actually get a number at the end.) In this case, switching the order of integration takes some work. We can't just say:

$$\cancel{\int_{-1}^{1}\int_{0}^{x^2+1} 1-x^2-y^2\,dy\,dx = \int_0^{x^2+1}\int_{-1}^{1} 1-x^2-y^2\,dx\,dy}$$

↑ variable on the outside!

② The bounds on the integral should never involve the dummy variable. This is an easy way to check for mistakes:

$$\int_0^{x^2+1} 1-x^2-y^2\,dx$$

doesn't make sense.

Thm: If D is a region given by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

This type of region is called y-Simple.

If D is a region given by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$ then

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\,dx\,dy$$

This type of region is called X-Simple.