Longo: Math 20C - Winter 2017
Lecture Notes

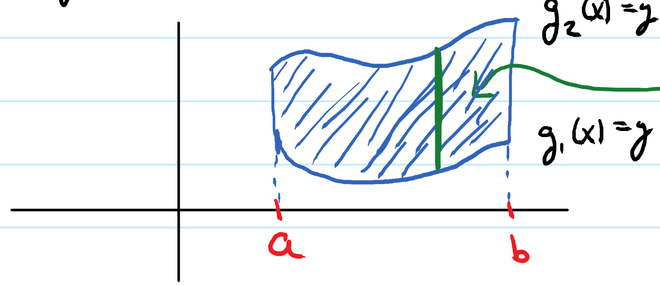
Date: March 15, 2017

Section:
- ☐ §5.3
- ☐ §5.4

Topics Covered:
- ☐ Double integrals over more general regions
- ☐ Changing the order of integration

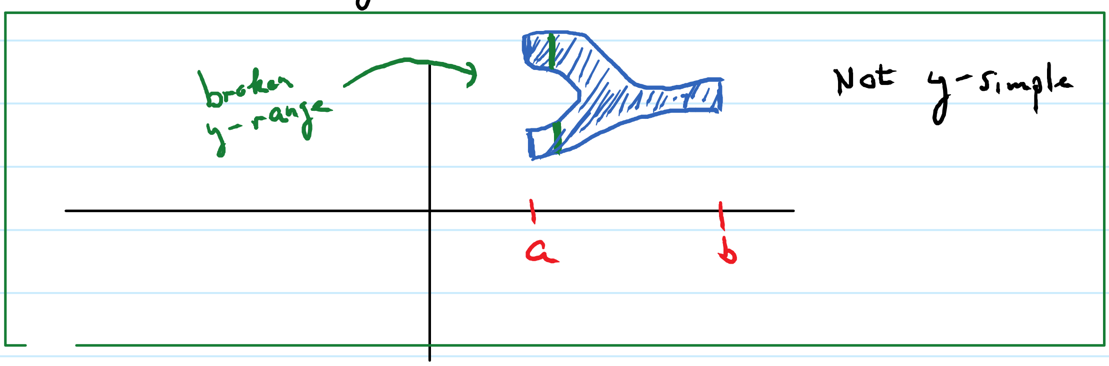
§ 5.3: Integrals over more general Regions:

Suppose we have a region, D, in the xy-plane that can be described as $D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$ where $g_1(x)$, $g_2(x)$ are fcns of x.



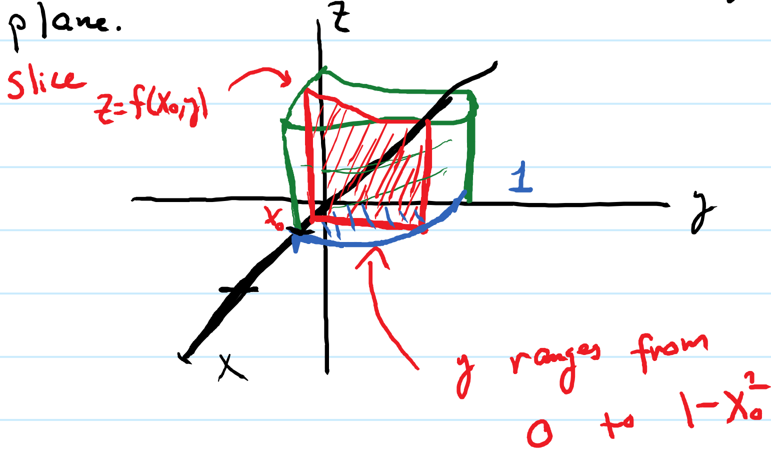
y-value is between $y = g_1(x)$ and $y = g_2(x)$.

This type of region is called y-simple because the bounds for y can be written as fcns of x.



Let f be a cont. fcn on a y-simple region D. (say $f(x,y) \ge 0$). Then we can still use Cavalieri's principle to calculate the volume under $z = f(x,y)$ and above the xy-plane.

Ex:



$$D = \left\{ (x,y) \;\middle|\; \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1 - x^2 \end{array} \right\}$$

$g_1(x)$ $\quad$ $g_2(x)$

For a fixed x-value, x_0, the area of a slice of the solid with $x = x_0$ is the area under the graph of $z = f(x_0, y)$ as $g(x) \le y \le g_1(x)$. If $A(x_0)$ is the area of a slice with x_0 fixed, then

$$A(x_0) = \int_{g_1(x_0)}^{g_2(x_0)} f(x_0, y)\, dy \qquad (x_0 \text{ constant!})$$

Therefore, the double integral $\iint_D f(x,y)\, dA$ can be expressed as an iterated integral by

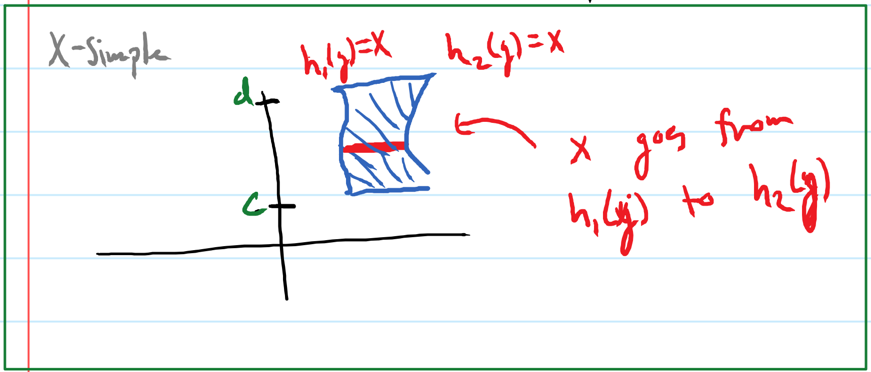
$$\iint_D f(x,y)\, dA = \int_a^b A(x)\, dx$$
$$= \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\, dy \right) dx.$$

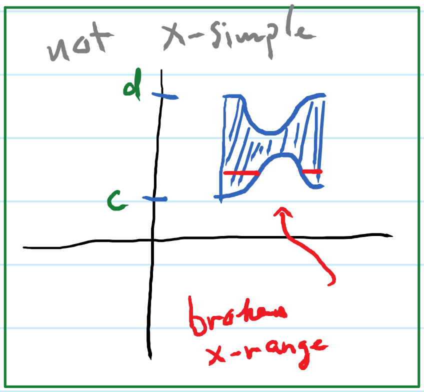
On the other hand, if D is an x-simple region. I.e., D can be described as

$$c \le y \le d, \quad h_1(y) \le x \le h_2(y)$$

(bounds for x are fcns of y). Then

$$\iint_D f(x,y)\, dA = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\, dx \right) dy$$





Example: Let $D = \left\{ (x,y) \;\middle|\; \begin{array}{l} 1 \le y \le 4 \\ \sqrt{y} \le x \le 4 \end{array} \right\}$

Calculate $\displaystyle\iint_D \frac{\sqrt{y}}{x^2}\, dA$.

Sol: Since D is written as an x-simple region, we should integrate w.r.t. x first:

$$\int_1^4 \int_0^{\sqrt{y}} \frac{\sqrt{y}}{x^2}\, dx\, dy$$

Warning: ① Outer integral Should not have a variable!
② In general, the bounds should never involve the dummy variable!

mistake

Common Mistake: ① $\displaystyle\iint_D \frac{\sqrt{y}}{x}\, dA = \int_{\sqrt{y}}^4 \int_1^4 \frac{\sqrt{y}}{x^2}\, dy\, dx$

② $\displaystyle\iint_D \frac{\sqrt{y}}{x}\, dA = \int_1^4 \int_{\sqrt{y}}^4 \frac{\sqrt{y}}{x}\, dy\, dx$

mistake

Inner integral (y-fixed): $\displaystyle\int_{\sqrt{y}}^4 \frac{\sqrt{y}}{x^2}\, dx$

$$= \sqrt{y} \int_{\sqrt{y}}^4 x^{-2}\, dx$$

$$= \sqrt{y} \left(-\frac{1}{x} \right) \Bigg|_{x=\sqrt{y}}^{x=4}$$

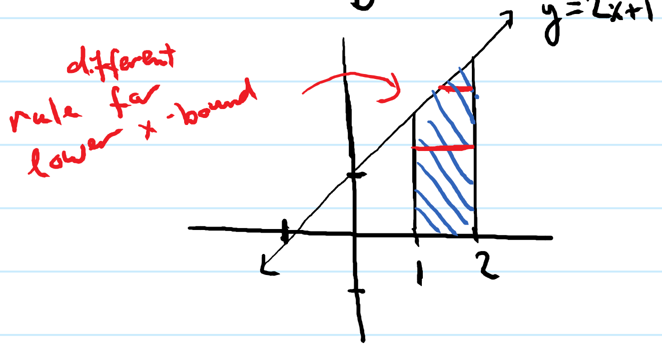
$$= \sqrt{y} \left(-\frac{1}{4} + \frac{1}{\sqrt{y}} \right)$$

$$= -\frac{\sqrt{y}}{4} + 1 .$$

$$\Rightarrow \iint_D \frac{\sqrt{y}}{x}\, dA = \int_1^4 \left(1 - \frac{\sqrt{y}}{4} \right) dy = \left(y - \frac{1}{4} \left(\frac{y^{3/2}}{\left(\frac{3}{2}\right)} \right) \right) \Bigg|_{y=1}^{y=4} = \left(y - \frac{1}{6} y^{\frac{3}{2}} \right) \Bigg|_{y=1}^{y=4}$$

$$\left(4-\tfrac{1}{6}(4)^{3/2}\right)-\left(1-\tfrac{1}{6}(1)^{3/2}\right) = \left(3+\tfrac{1+8}{6}\right)$$
$$= 3+\tfrac{3}{2}$$
$$= \frac{9}{2}$$

Ex: Calculate $\iint_D e^{x^2+x}\,dA$ where D is the region bound by $x=1$, $x=2$, $y=0$, $y=2x+1$.



Note: D is not x-simple since if choose different y-values, the "rule" for the bound changes. D is y simple since for any x-value,

$$0 \le y \le 2x+1.$$

The overall range for x on D is $1 \le x \le 2$.
Therefore, we get the iterated integral:

$$\iint_D e^{x^2+x}\,dA = \int_1^2 \int_0^{2x+1} e^{x^2+x}\,dy\,dx$$

Note: It's a good thing we're integrating w.r.t. y first since $\int e^{x^2+x}dx$ is not possible with "elementary" techniques.

Inner Integral: (x-fixed)
$$\int_0^{2x+1} e^{x^2+x}\,dy = \left(y e^{x^2+x}\right)\Big|_{y=0}^{y=2x+1}$$
$$= \boxed{(2x+1)e^{x^2+x}}$$

Plugging back into the outer fcn, we get

$$\int_1^2 \underline{(2x+1)} e^{x^2+x}\, dx \quad . \quad \text{Let } u = x^2+x \Rightarrow du = (2x+1)dx$$

now we can substitute!

$$\Rightarrow \int_1^2 (2x+1) e^{x^2+x}\, dx$$

$$= \int_{u(1)}^{u(2)} e^u\, du.$$

$$= \left(e^u\right)\Big|_{u(1)}^{u(2)}$$

$$= \left(e^{x^2+x}\right)\Big|_1^2 = \boxed{e^6 - e^2}$$

In this example, we saw that we could not integrate w.r.t. x in the original fcn. However, once we integrated w.r.t y, the integral became possible. This example illustrates why we might want to change the order of integration.

We will see how to do this next time.