Longo: Math 20C - Winter 2017
Lecture Notes

Date: March 17, 2017

Section:

- [ ] §5.4

Topics Covered:

- [ ] Changing the order of integration

§5.4: Changing the Order of Integration:

Last time, we saw that sometimes it may be beneficial to change the order of integration in order to have a reasonable integrand. Let's formalize this idea.

Let D be a region in the xy-plane that is both x-simple and y-simple. I.e., D can be written in the form

$$D = \left\{(x,y) \,\middle|\, \begin{matrix} c \le y \le d \\ h_1(y) \le x \le h_2(y) \end{matrix}\right\} \longleftrightarrow \text{(x-simple)}$$

and

$$D = \left\{(x,y) \,\middle|\, \begin{matrix} a \le x \le b \\ g_1(x) \le y \le g_2(x) \end{matrix}\right\} \longleftrightarrow \text{(y-simple)}$$

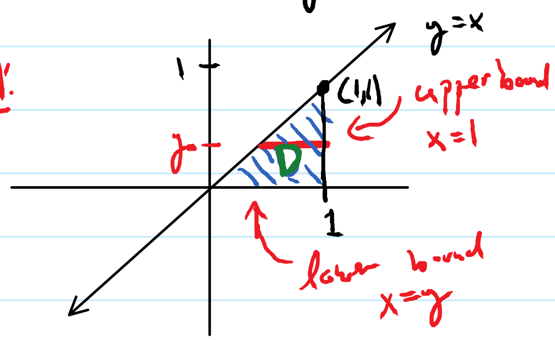
Rmk: We call such a domain simple.

If $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is continuous, then we can write $\iint_D f\,dA$ as an iterated integral in two different ways:

$$\iint_D f\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Ex: Let D be the domain bounded by $x=0$, $x=1$, $y=0$, $y=x$. If $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is continuous, express $\iint_D f\,dA$ as an iterated integral in two different ways.

Sol:

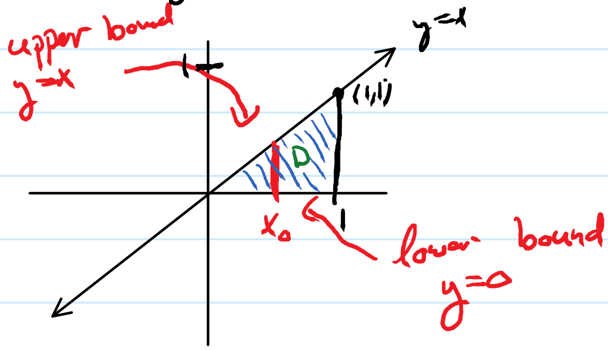


As an x-simple domain, if we fix a y-value, x is bounded from below by $x=y$, and x is bounded from above by $x=1$.

meanwhile, the largest y-value on D is 1, and the smallest y-value on D is 0. So as an x-simple domain,

$$D=\left\{(x,y)\,\middle|\, \begin{array}{l} 0\le y\le 1 \\ y\le x\le 1 \end{array}\right\}. \Rightarrow \boxed{\iint_D f\,dA=\int_0^1\int_y^1 f(x,y)\,dx\,dy.}$$

As a y-simple domain, if we fix an x-value, then y ranges from $y=0$ to $y=x$.



Meanwhile, the smallest x-value on D is 0 while the largest is 1. So as a y-simple domain,

$$D=\left\{(x,y)\,\middle|\, \begin{array}{l} 0\le x\le 1 \\ 0\le y\le x \end{array}\right\}\Rightarrow$$

$$\boxed{\iint_D f\,dA=\int_0^1\int_0^x f(x,y)\,dy\,dx}$$

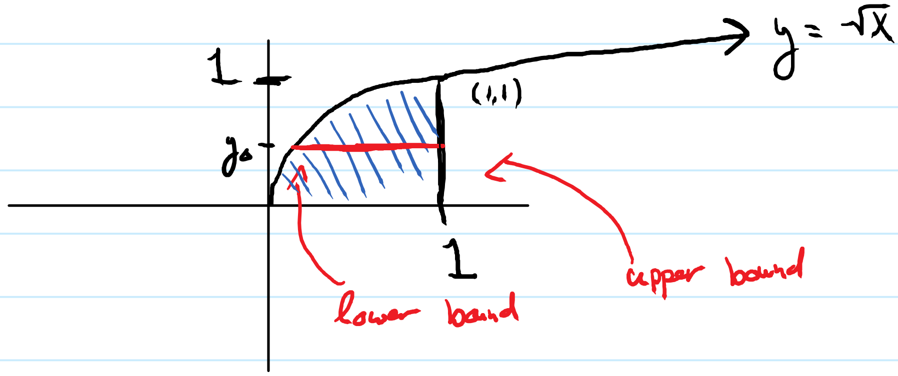
Warning: A common mistake is to simply replace every x with a y in the bounds (or vice versa).

$$\int_0^1\int_0^x f(x,y)\,dy\,dx \neq \int_0^1\int_0^y f(x,y)\,dx\,dy$$

Ex: Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{x}} x\, e^{y^5-5y}\,dy\,dx$

Sol: Unfortunately, we don't have an "elementary" way to evaluate the integral $\int_0^{\sqrt{x}} x\, e^{y^5-5y}\,dy$. Let's switch the order of integration.

Step 1: (Draw the domain): We are integrating over the region $D\subseteq\mathbb{R}^2$ defined by: $0\le x\le 1$
$0\le y\le \sqrt{x}$



To express D as an X-simple domain, we fix a y value and find the range for x in terms of y. For a fixed y-value, the smallest x can be is when x is on the curve $y=\sqrt{x} \Rightarrow \boxed{x=y^2}$. The largest x value is 1. Meanwhile, the absolute bounds for y on D are $0 \le y \le 1$. Therefore, we have

$0 \le y \le 1$, $y^2 \le x \le 1$. By switching the order of integration, we get

$$\int_0^1\int_0^{\sqrt{x}} x e^{y^5-5y}\,dy\,dx = \int_0^1\int_{y^2}^1 x e^{y^5-5y}\,dx\,dy$$

Now we can evaluate the inner integral:

$$\int_{y^2}^1 x e^{y^5-5y}\,dx \quad \left(\frac{x^2}{2} e^{y^5-5y}\right)\Big|_{x=y^2}^{x=1} = \left(\frac{1}{2}-\frac{(y^2)^2}{2}\right)e^{y^5-5y}$$

$$= \frac{1}{2}(1-y^4)e^{y^5-5y}$$

$$\Rightarrow \int_0^1\int_{y^2}^1 x e^{y^5-5y}\,dx\,dy = \int_0^1 \frac{1}{2}(1-y^4)e^{y^5-5y}\,dy$$

Now we make the substitution, $u=y^5-5y$ $(\Rightarrow u(0)=0,\ u(1)=-4$

$$\Rightarrow du=(5y^4-5)\,dy$$
$$\Rightarrow -\frac{1}{5}du=(1-y^4)\,dy$$

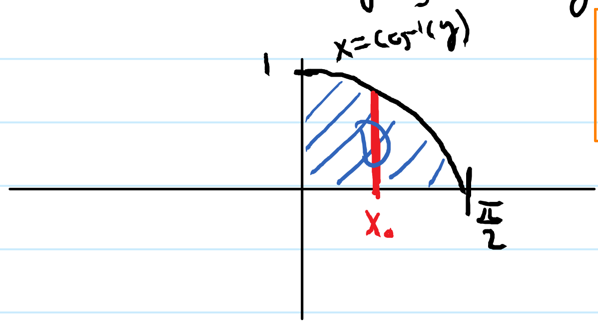
$$\Rightarrow \int_0^1 \frac{1}{2}(1-y^4)e^{y^5-5y}\,dy = -\frac{1}{10}\int_0^{-4} e^u\,du = -\frac{1}{10}(e^{-4}-e^0)$$

$$= -\frac{1}{10}(e^{-4}-1)$$

Ex: Evaluate $\displaystyle\int_0^1\int_0^{\cos^{-1}(y)} \sqrt{1+\sin(x)}\,dx\,dy$

Since $\int \sqrt{1+\sin(x)}\,dx$ isn't obvious, let's change the order of integration. Here, the domain is written as:

$0 \le x \le \cos^{-1}(y)$, $0 \le y \le 1$



$x=\cos^{-1}(y) \Leftrightarrow y=\cos(x)$ and $0 \le x \le \pi$

} By definition of $f(y)=\cos^{-1}(y)$!

We need to fix an x-value and solve for y in terms of x. Since $x=\cos^{-1}(y) \Leftrightarrow y=\cos(x)$ & $x\in[0,\pi]$, the upper bound for y in terms of x is $y=\cos(x)$, and the lower bound is $y=0$. $\Rightarrow$ $\boxed{0 \le y \le \cos(x).}$

Since $x=\cos^{-1}(y)$, and $y=0$ intersect when $x=\frac{\pi}{2}$, the absolute bounds for x are $\boxed{0 \le x \le \frac{\pi}{2}.}$

Therefore, $\displaystyle\int_0^1\int_0^{\cos^{-1}(y)} \sqrt{1+\sin(x)}\,dx\,dy = \int_0^{\frac{\pi}{2}}\int_0^{\cos(x)} \sqrt{1+\sin(x)}\,dy\,dx$

$$= \int_0^{\frac{\pi}{2}} \left(y\sqrt{1+\sin(x)} \right)\Bigg|_{y=0}^{y=\cos(x)} dx$$

$$= \int_0^{\pi/2} \cos(x)\sqrt{1+\sin(x)}\,dx.$$

Now we make the substitution

$u=1+\sin(x) \Rightarrow$
- $u(0)=1$
- $u(\frac{\pi}{2})=2$
- $du=\cos(x)dx$

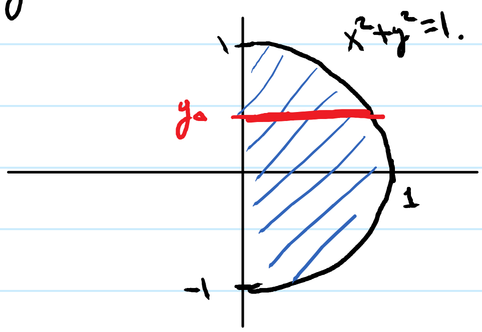
- $du = \cos(x)\,dx$

$$\int_0^{\frac{\pi}{2}} \cos(x)\sqrt{1+\sin(x)}\,dx = \int_1^2 \sqrt{u}\,du$$
$$= \left(\frac{2}{3}u^{3/2}\right)\Big|_{u=1}^{u=2}$$
$$= \boxed{\frac{2}{3}\left(2^{3/2}-1\right)}$$

Example: Evaluate $\displaystyle\int_0^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{x}{\sqrt{(x^2+y^2)}}\,dy\,dx.$

Sol: Let D be the domain of integration. Then D can be described by $\{(x,y) \mid -\sqrt{1-x^2} \le y \le \sqrt{1-x^2},\ 0 \le x \le 1\}$.

Note: the fcns $y=\sqrt{1-x^2}$, $y=-\sqrt{1-x^2}$ describe the upper and lower hemispheres of the unit circle. Since $0 \le x \le 1$, D is the right half of the unit disc.



To switch the order of integration, we need to fix a y-value and express the bounds of x in terms of y. If y is fixed, the lower bound for x is 0 and the upper bound for x is when the point hits the unit circle $x^2+y^2=1 \Rightarrow x=\sqrt{1-y^2}$ (only positive since we already knew $x \ge 0$.)

Meanwhile the absolute bounds for y on D are

$$D = \left\{(x,y) \;\middle|\; \begin{array}{c} -1 \le y \le 1 \\ 0 \le x \le \sqrt{1-y} \end{array}\right\}$$

$$\Rightarrow \int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{x}{(x^2+y^2)^{1/2}}\, dy\, dx = \int_{-1}^{1} \int_0^{\sqrt{1-y^2}} \frac{x}{(x^2+y^2)^{1/2}}\, dx\, dy.$$

<u>Inner Integral:</u> (y is const.)

$$\int_0^{\sqrt{1-y^2}} \frac{x}{(x^2+y^2)^{1/2}}\, dx.$$

Let $u = x^2+y^2 \Rightarrow$

- $u(0) = y^2$
- $u(\sqrt{1-y^2}) = (\sqrt{1-y^2})^2 + y^2 = 1$
- $du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx.$

$$\int_0^{\sqrt{1-y^2}} \frac{x}{(x^2+y^2)^{1/2}}\, dx = \frac{1}{2}\int_{y^2}^{1} \frac{1}{u^{1/2}}\, dx = \frac{1}{2}\int_{y^2}^{1} u^{-\frac{1}{2}}\, du$$

$$= \frac{1}{2}\left(2u^{\frac{1}{2}}\right)\Big|_{u=y^2}^{u=1}$$

$$= \frac{1}{2}\left(2 + 2(y^2)^{1/2}\right)$$

$$= 1+y.$$

<u>Outer integral:</u>

$$\int_{-1}^{1}\int_0^{\sqrt{1-y^2}} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy = \int_{-1}^{1} 1+y\, dy = \left(y + \tfrac{1}{2}y^2\right)\Big|_{-1}^{1}$$

$$= \left(1+\tfrac{1}{2}\right) - \left(-1+\tfrac{1}{2}\right)$$

$$= \boxed{2}$$