Longo: Math 20C - Winter 2017
Lecture Notes

Date: January 13, 2017

Section:
- [ ] §1.2 (cont.)

Topics Covered:
- [ ] Unit vectors and normalization
- [ ] Geometry of the inner product and orthogonality
- [ ] Orthogonal projections

§1.2 (cont.):

Recall from last time: If $\vec{u} = (x_0, y_0, z_0)$ is a vector in $\mathbb{R}^3$. $\boxed{\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{x_0^2 + y_0^2 + z_0^2}}$

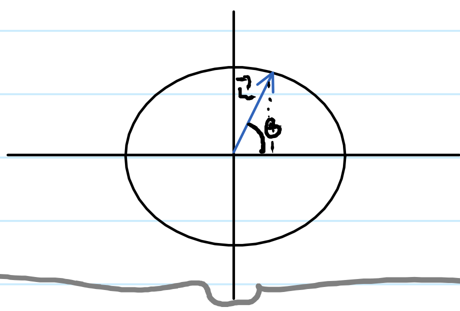
For vectors in $\mathbb{R}^2$, just ignore the z-component.

Unit vectors: A vector $\vec{u}$ is called a unit vector iff $\|\vec{u}\| = 1$.

Remark: The word "unit" comes from the latin word 'unum' meaning 1.

Example: (Unit Vectors in $\mathbb{R}^2$:). Suppose $\vec{u} = (x_0, y_0)$ is a unit vector. Then $x_0^2 + y_0^2 = 1$.

Therefore, if we put the tail of $\vec{u}$ at the origin, then the head is on the unit circle.



If θ is the angle between $\vec{u}$ and the positive real axis, then $\boxed{\vec{u} = (\cos\theta, \sin\theta)}$.

Let $\vec{u}$ be any vector. If we multiply $\vec{u}$ by the scalar $\frac{1}{\|\vec{u}\|}$, it "shrinks" $\vec{u}$ by a factor of $\|\vec{u}\|$. The resulting vector, $\boxed{\vec{e}_{\vec{u}} = \frac{1}{\|\vec{u}\|}\vec{u}}$, has length 1. This process is called normalization.

Example: Normalize $\vec{u} = (-6, 1, 9)$.

Calculate: $\|\vec{u}\| = \sqrt{(-6)^2 + (1)^2 + (9)^2} = \sqrt{118}$

so

$$\boxed{\begin{aligned} e_{\vec{u}} &= \frac{1}{\sqrt{118}}(-6, 1, 9) \\ &= \left(\frac{-6}{\sqrt{118}}, \frac{1}{\sqrt{118}}, \frac{9}{\sqrt{118}}\right) \end{aligned}}$$

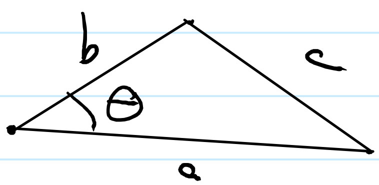
Geometry of the inner product:

We haven't yet discussed why we use the dot product. The summary is:

$\boxed{\text{The dot product tells you about the (smallest) angle between two vectors.}}$

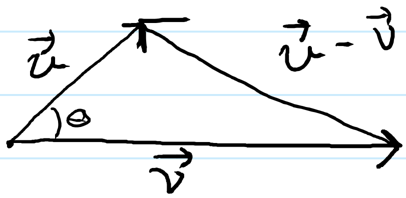
Let's see how. First we need the

Law of Cosines: In the triangle:



we have $a^2+b^2=c^2+2ab\cos\theta$

Let $\vec{u}, \vec{v}$ be two vectors. Draw them so that they start at a common point, and draw the difference vector, $\vec{u}-\vec{v}$. Let θ be the smallest angle between $\vec{u},\vec{v}$.



By the law of cosines:

$$(*)\quad \|\vec{u}\|^2+\|\vec{v}\|^2=\|\vec{u}-\vec{v}\|^2+2\|\vec{u}\|\|\vec{v}\|\cos\theta.$$

Since
$$\begin{aligned}\|\vec{u}-\vec{v}\|^2 &= (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})\\ &= \vec{u}\cdot(\vec{u}-\vec{v})-\vec{v}\cdot(\vec{u}-\vec{v})\\ &= \vec{u}\cdot\vec{u}\underbrace{-\vec{u}\cdot\vec{v}-\vec{v}\cdot\vec{u}}+\vec{v}\cdot\vec{v}\\ &= \|\vec{u}\|^2-2(\vec{u}\cdot\vec{v})+\|\vec{v}\|^2.\end{aligned}$$

So eqn (*) becomes

$$\|\vec{u}\|^2 + \|\vec{v}\|^2 = \|\vec{u}\|^2 - 2(\vec{u}\cdot\vec{v}) + \|\vec{v}\|^2 + 2\|\vec{u}\|\,\|\vec{v}\|\cos\theta.$$

$$\Rightarrow \quad 0 = -2(\vec{u}\cdot\vec{v}) + 2\|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

$$\Rightarrow \quad \boxed{\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta}$$

Important!

Example: What is the smallest angle, θ, between $\vec{u} = (-1, 0, 1)$, $\vec{v} = (7, -13, 4)$.

Sol: The above equation implies:

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right).$$

Calculate:

$$\boxed{\vec{u}\cdot\vec{v} = (-1)(7) + (0)(-13) + (1)(4) = -3}$$

$$\boxed{\|\vec{u}\| = \sqrt{(-1)^2 + (0)^2 + (1)^2} = \sqrt{2}}$$

$$\boxed{\|\vec{v}\| = \sqrt{(7)^2 + (-13)^2 + 4^2} = \sqrt{49 + 169 + 16} = \sqrt{234}}$$

So $$\boxed{\theta = \cos^{-1}\left(\frac{-3}{\sqrt{2}\cdot\sqrt{236}}\right).}$$

Remarks: Since $\|\vec{u}\|, \|\vec{v}\| > 0$,

① $\vec{u}\cdot\vec{v} > 0 \Leftrightarrow \cos\theta > 0 \Leftrightarrow 0 \leq \theta < \frac{\pi}{2}$ (θ is acute)

② $\vec{u}\cdot\vec{v} < 0 \Leftrightarrow \cos\theta < 0 \Leftrightarrow \frac{\pi}{2} < \theta \leq \pi$ (θ is obtuse)

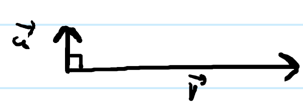
and most importantly,

③ $\vec{u}\cdot\vec{v} = 0$ iff $0 = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$

$\vec{u}\cdot\vec{v} = 0$ iff $\cos\theta = 0$

$$\boxed{\vec{u}\cdot\vec{v} = 0 \quad \text{iff} \quad \theta = \frac{\pi}{2} \quad (= 90^\circ).}$$

Important



Since the dot product is so easy to calculate this is a very quick test to see if two vectors meet at a right angle.

Terminology: Two vectors $\vec{u}, \vec{v}$ are said to be orthogonal or normal or perpendicular if they meet at a right angle (iff $\vec{u}\cdot\vec{v} \stackrel{?}{=} 0$). We denote this by $\vec{u} \perp \vec{v}$.

Example: Decide if the angle between $\vec{u} = (-1, 1, 7)$, $\vec{v} = (-3, 10, 5)$ is obtuse, accute, or right.

$\boxed{\text{Sol:}}$ Use the dot product!

$$\vec{u}\cdot\vec{v} = (-1)(-3) + (1)(10) + (7)(5) > 0$$

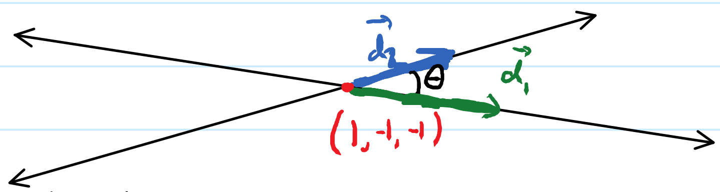
so the angle between $\boxed{\vec{u}, \vec{v} \text{ is acute.}}$

Example: Prove that the lines

$$\ell_1(t) = (1, -1, -1) + t(-5, 2, 2)$$
$$\ell_2(t) = (1, -1, -1) + t(6, 9, 6)$$

intersect at a right angle.

$\boxed{\text{Sol:}}$ We can tell by inspection that the point $(1,-1,-1)$ is on both lines, so they definitely intersect. Notice that to find the angle between ℓ_1, ℓ_2, we need to find the angle between their direction vectors $\vec{d_1} = (-5, 2, 2)$, $d_2 = (6, 9, 6)$.

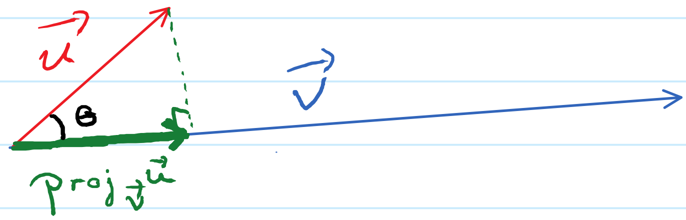


Note: $\vec{d_1}\cdot\vec{d_2} = (-5)(6) + (2)(9) + (2)(6) = -30 + 18 + 12 = 0.$

Therefore $\ell_1 \perp \ell_2$.

Orthogonal Projections:

Start with two vectors $\vec{u}, \vec{v}$.



Imagine you shine a flashlight straight down onto $\vec{v}$. The shadow that $\vec{u}$ casts onto $\vec{v}$ is called the projection of $\vec{u}$ along (or onto) $\vec{v}$. It is denoted $\text{proj}_{\vec{v}}\vec{u}$.

To determine $\text{proj}_{\vec{v}}\vec{u}$, we just need its direction and magnitude:

Using basic trigonometry, we see $\cos(\theta) = \dfrac{\|\text{proj}_{\vec{v}}\vec{u}\|}{\|\vec{u}\|}$

$$\Rightarrow \boxed{\|\text{proj}_{\vec{v}}\vec{u}\| = \|\vec{u}\|\cos\theta = \frac{|\vec{u}\cdot\vec{v}|}{\|\vec{v}\|}.}$$

Secondly, $\text{proj}_{\vec{v}}\vec{u}$ points in the direction of $\vec{v}$. So the unit vector that points in the direction of $\text{proj}_{\vec{v}}\vec{u}$ is $\pm\dfrac{\vec{v}}{\|\vec{v}\|}$.

If we are careful with the $\pm$ sign we see

$$\text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}\right)\frac{\vec{v}}{\|\vec{v}\|}$$

$$= \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v}$$

$$= \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$$

Example: Find the projection of $\vec{u} = \langle 9, -4 \rangle$ onto $\vec{v} = \langle 2, 2 \rangle$

Sol:

$$\vec{u} \cdot \vec{u} = (9)(2) + (-4)(2)$$
$$= 18 - 8 = 10$$
$$\vec{v} \cdot \vec{v} = (2)(2) + (2)(2) = 8$$

$$\text{proj}_{\vec{v}} \vec{u} = \left(\frac{\vec{u} \cdot \vec{u}}{\vec{v} \cdot \vec{v}} \right) \vec{v} = \frac{10}{8} \langle 2, 2 \rangle$$
$$= \frac{5}{4} \langle 2, 2 \rangle$$
$$= \left\langle \frac{5}{2}, \frac{5}{2} \right\rangle$$