# Longo: Math 20C - Winter 2017
# Lecture Note

Date: January 18,2017

Section:
- [ ] §1.2 (cont.)
- [ ] §1.3

Topics Covered:
- [ ] Orthogonal projections
- [ ] Determinants, area and volume

Since the dot product is so easy to calculate this is a very quick test to see if two vectors meet at a right angle.

**Terminology:** Two vectors $\vec{u}, \vec{v}$ are said to be **orthogonal** or **normal** or **perpendicular** if they meet at a right angle (iff $\vec{u}\cdot\vec{v} \stackrel{?}{=} 0$). We denote this by $\vec{u} \perp \vec{v}$.

**Example:** Decide if the angle between $\vec{u} = (-1, 1, 7)$, $\vec{v} = (-3, 10, 5)$ is obtuse, accute, or right.

$\boxed{\text{Sol:}}$ Use the dot product!

$$\vec{u}\cdot\vec{v} = (-1)(-3) + (1)(10) + (7)(5) > 0$$

so the angle between $\vec{u}, \vec{v}$ is acute.

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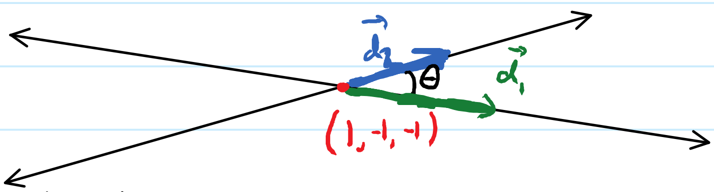
**Example:** Prove that the lines

$$\ell_1(t) = (1,-1,-1) + t(-5,2,2)$$
$$\ell_2(t) = (1,-1,-1) + t(6,9,6)$$

intersect at a right angle.

$\boxed{\text{Sol:}}$ We can tell by inspection that the point $(1,-1,-1)$ is on both lines, so they definitely intersect.

Notice that to find the angle between $\ell_1, \ell_2$, we need to find the angle between their direction vectors $\vec{d_1} = (-5, 2, 2)$, $d_2 = (6, 9, 6)$.

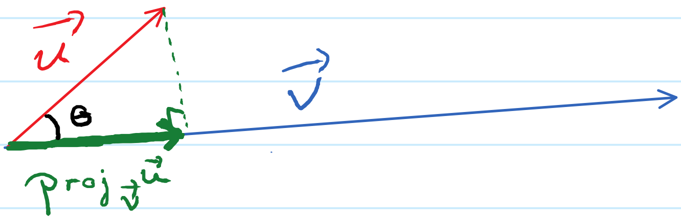



**Note:** $\vec{d_1}\cdot\vec{d_2} = (-5)(6) + (2)(9) + (2)(6) = -30 + 18 + 12 = 0$.

Therefore $\ell_1 \perp \ell_2$.

# Orthogonal Projections:

Start with two vectors $\vec{u}, \vec{v}$.




Imagine you shine a flashlight straight down onto $\vec{v}$. The shadow that $\vec{u}$ casts onto $\vec{v}$ is called the projection of $\vec{u}$ along (or onto) $\vec{v}$. It is denoted $\text{proj}_{\vec{v}}\vec{u}$.

To determine $\text{proj}_{\vec{v}}\vec{u}$, we just need its direction and magnitude:

Using basic trigonometry, we see $\cos(\theta) = \dfrac{\|\text{proj}_{\vec{v}}\vec{u}\|}{\|\vec{u}\|}$

$$\Rightarrow \boxed{\|\text{proj}_{\vec{v}}\vec{u}\| = \|\vec{u}\|\cos\theta = \frac{|\vec{u}\cdot\vec{v}|}{\|\vec{v}\|}.}$$

Secondly, $\text{proj}_{\vec{v}}\vec{u}$ points in the direction of $\vec{v}$. So the unit vector that points in the direction of $\text{proj}_{\vec{v}}\vec{u}$ is $\pm\dfrac{\vec{v}}{\|\vec{v}\|}$.

If we are careful with the $\pm$ sign we see

$$\boxed{\begin{aligned}\text{proj}_{\vec{v}}\vec{u} &= \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}\right)\frac{\vec{v}}{\|\vec{v}\|} \\ &= \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v} \\ &= \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}\end{aligned}}$$

**Example:** Find the projection of $\vec{u} = (9, -4)$ onto $\vec{v} = (2, 2)$

$\boxed{\text{Sol:}}$

$$\vec{u} \cdot \vec{u} = (9)(2) + (-4)(2)$$
$$= 18 - 8 = 10$$
$$\vec{v} \cdot \vec{v} = (2)(2) + (2)(2) = 8$$

$$\boxed{\begin{aligned} \text{proj}_{\vec{v}} \vec{u} = \left(\frac{\vec{u} \cdot \vec{u}}{\vec{v} \cdot \vec{v}}\right) \vec{v} &= \frac{10}{8}(2,2) \\ &= \frac{5}{4}(2,2) \\ &= \left(\frac{5}{2}, \frac{5}{2}\right) \end{aligned}}$$

## § 1.3: The cross Product:

The immediate goal is to define the second way we can "multiply" two vectors. We will eventually use it to come up with equations of planes. First, we revew determinants.

Let $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ be a $2 \times 2$ matrix. The determinant of $A$, denoted $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ or $\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$, is defined by:

$$\boxed{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1}$$

**Example:** $\begin{vmatrix} 4 & 7 \\ -3 & 10 \end{vmatrix} = (4)(10) - (7)(-3) = 40 + 21 = \boxed{61}$

Rmk: We will see shortly that the determinant of a matrix carries information about area/volume:

**Q!** What about $3\times3$ matrices?

Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$. To find the determinant, we "expand along the first row:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

To get this, circle $a_1$ and cross out its row and column. The first term is $a_1 \times$ (the determinant of the $2\times2$ matrix that is not crossed out). Then move onto $a_2$ and do the same thing, except now there is a negative sign. Then move onto $a_3$ but with a + sign.

**Example:**

$$\begin{vmatrix} 1 & -1 & 2 \\ -3 & 0 & 4 \\ 9 & -5 & -5 \end{vmatrix} = 1 \begin{vmatrix} 0 & 4 \\ -5 & -5 \end{vmatrix} - (-1) \begin{vmatrix} -3 & 4 \\ 9 & -5 \end{vmatrix} + 2 \begin{vmatrix} -3 & 0 \\ 9 & -5 \end{vmatrix}$$

$$= 1\big((0)(-5) - (4)(-5)\big) - (-1)\big((-3)(-5) - (4)(9)\big) + 2\big((-3)(-5) - (0)(9)\big)$$
$$= 1(20) + 1(15-36) + 2(15)$$
$$= 20 - 21 + 30$$
$$= \boxed{29}$$

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## The Cross Product:

We use the determinant to define the "cross product" of two vectors in $\mathbb{R}^3$.

**Def:** Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$ be two vectors in $\mathbb{R}^3$. The *cross product* of $\vec{u}$ and

$\vec{v}$, denoted by $\vec{u}\times\vec{v}$, is defined by:

$$\boxed{\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\vec{k}}$$

Example: Let $\vec{u}=(5,5,0)$, $\vec{v}=(-1,1,1)$.

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 5 & 0 \\ -1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 5 & 0 \\ 1 & 1 \end{vmatrix}\vec{i} - \begin{vmatrix} 5 & 0 \\ -1 & 1 \end{vmatrix}\vec{j} + \begin{vmatrix} 5 & 5 \\ -1 & 1 \end{vmatrix}\vec{k}$$

$$= 5\vec{i} - 5\vec{j} + 10\vec{k}$$

$$\boxed{= (5, -5, 10)}$$

First Properties: 1 Anticommutativity: $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$

2 $(k\vec{u})\times\vec{v} = k(\vec{u}\times\vec{v}) = \vec{u}\times(k\vec{v})$.

3 $(\vec{u}+\vec{v})\times\vec{w} = (\vec{u}\times\vec{w}) + (\vec{v}\times\vec{w})$

$\vec{w}\times(\vec{u}+\vec{v}) = (\vec{w}\times\vec{u}) + (\vec{w}\times\vec{v})$

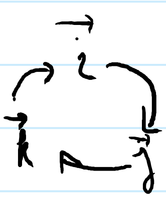
Rmk: 1: Prop. 1 says the order that you put $\vec{u}, \vec{v}$ into the matrix matters, in $\vec{u}\times\vec{v}$, $\vec{u}$ goes in row 2, $\vec{v}$ goes in row 3.

2 These are proved using basic calculations. Props. 1, 2 also come from basic determinant properties.

Fact: Easy calculations show:

$$\boxed{\begin{aligned} \vec{i}\times\vec{j} &= \vec{k} \\ \vec{j}\times\vec{k} &= \vec{i} \\ \vec{k}\times\vec{i} &= \vec{j} \end{aligned}}$$

An easy way to remember this is the diagram:




Next time we discuss the geometry of the cross product.